An Interest Rate Swap Volatility Index and Contract

Antonio Mele — *QUASaR*

Yoshiki Obayashi — *Applied Academics LLC*

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ABSTRACT Interest rate volatility and equity volatility evolve heterogeneously over time, co-moving disproportionately during periods of global imbalances and each reacting to events of different nature. While the Chicago Board Options Exchange® (CBOE®) VIX® reflects the fair value of equity volatility, no interest rate counterparts exist to the CBOE VIX®; we fill this gap for swap rate volatility. We use data on interest rate swaptions and bonds to construct two indexes of interest rate swap volatility expected to prevail in a risk-neutral market within any given investment horizon. The two indexes match market practices of quoting swaption-implied volatilities both in terms of basis point and percentage volatility. The indexes are constructed such that they reflect the model-free fair value of variance swap contracts for forward swap rates that we design to account for the uncertainty in the annuity factor of the fixed leg of forward swaps. The indexes are model-free in the sense that they do not rely on any particular option pricing model. While the end-result can be thought of as the swap rate counterpart to the CBOE VIX® for equity volatility, the mathematical formulations, contract designs, replication arguments, and derivatives underlying the index are materially different. The differences arise as we must account not only for changes in forward swap rates but also in the yield curve points impacting forward swap values in a framework where interest rates are not assumed to be constant as they are in the case of the VIX® methodology. We also consider a framework in which swap rates may experience jumps and find that our basis point volatility index can be expressed in a model-free format even in the presence of discontinuities.

Contents

1. Introduction

Interest rate volatility and equity volatility evolve heterogeneously over time, comoving disproportionately during periods of global imbalances and each reacting to events of different nature. While the Chicago Board Options Exchange® (CBOE®) VIX® reflects the fair value of equity volatility, no interest rate counterparts exist to the CBOE VIX®; we fill this gap for swap rate volatility.

Figure 1 depicts the 10 year spot swap rate over nearly two decades and compares its 20 day realized volatility with that of the S&P 500® Index. While these two volatilities share some common trends and spikes, one is hardly a proxy for the other. Their average correlation over the sample size is a mere 51%. Moreover, Figure 2 shows that the correlation between the two volatilities experiences large swings over time. For instance, before the financial crisis starting in 2007, the correlation was negative. In the first years of this crisis, the same correlation was high, most likely due to global concerns about disorderly tail events. However, even during the crisis, we see periods of marked divergences between the two volatilities. After 2010, for example, the correlation between the two volatilities plummeted, reaching negative values in 2011.

How can one hedge against or formulate views on uncertainties around changes in swap rates? In swap markets, which are the focus of this paper, volatility is typically traded through swaption-based strategies such as straddles: for example, going long both a payer and a receiver swaption may lead to profits should interest rates experience a period of high volatility. However, delta-hedged or unhedged swaption straddles do not necessarily lead to profits consistent with directional volatility views. The reason for this discrepancy is, at least partially, similar to that explaining the failure of equity option straddles to deliver P&L consistent with directional volatility views. Straddles suffer from "price dependency," the tendency to generate P&L affected by the direction of price movements rather than their absolute movements—i.e. volatility. These, and related reasons, have led to variance swap contracts in equity markets designed to better align volatility views and payoffs, and a new VIX® index maintained by the Chicago Board Options Exchange since 2003.

Despite the fact that transaction volumes in the swap and swaption markets are orders of magnitude greater than in equity markets, no such index or variance contracts exist. Moreover, there are only weak linkages between interest rate swap volatility and equity volatility, as illustrated by Figures 1 and 2, which further motivates the creation of a VIX-like index for swap rates. At the same time, the complexity of interest rate transactions is such that their volatility is more difficult to price. Intuitively, one prices assets by discounting their future payoffs, but when it comes to fixed income securities, discounting rates are obviously random and, in turn, interest rate swap volatility depends on this randomness. An additional critical issue pertaining to swap rate volatility is that uncertainty in swap markets is driven by two sources of randomness: first, a direct source relating to changes in swap rates; and second, an indirect one pertaining to changes in points of the yield curve that affect the value of swap contracts.

This paper introduces security designs to trade uncertainty related to developments of interest rate swap volatility while addressing the above issues. The pricing of these products is model-free, as it only relies on the price of traded assets such as (i) European-style swaptions, serving as

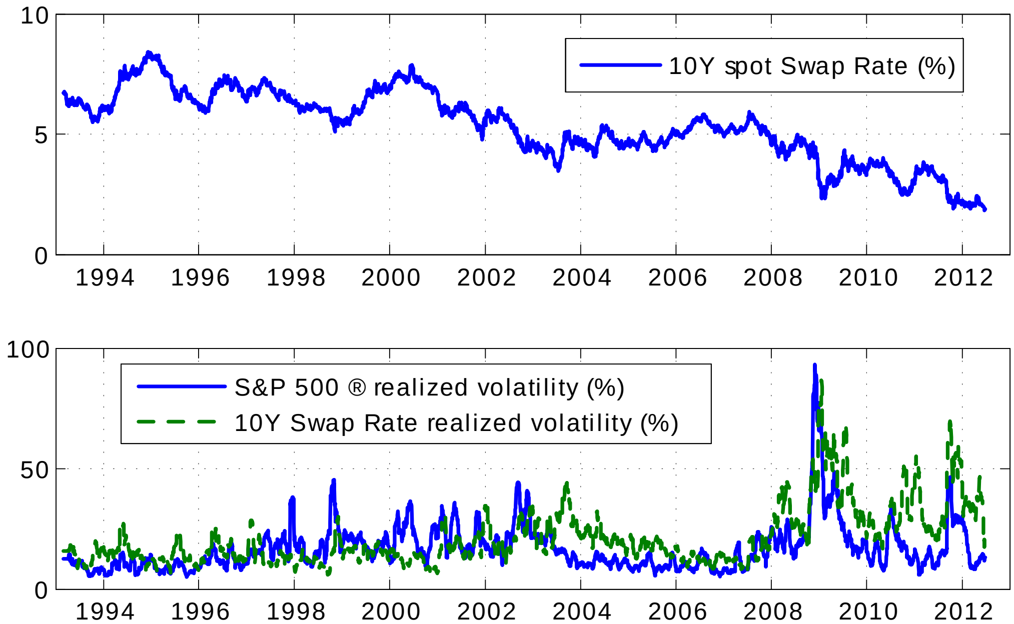

FIGURE 1. Top panel: The 10 year Swap Rate, annualized, percent. Bottom panel: estimates of 20 day realized volatilities of (i) the S&P 500® daily returns (solid line), and (ii) the daily logarithmic changes in the 10 year Swap Rate, annualized, percent, $100\sqrt{12\sum_{t=1}^{21}\ln^2\frac{\mathrm{X}_{t-i+1}}{\mathrm{X}_{t-i}}}$, where X_t denotes either the S&P 500 Index® or the 10 year Swap Rate. The sample includes daily data from January 29, 1993 to May 22, 2012, for a total of 4865 observations.

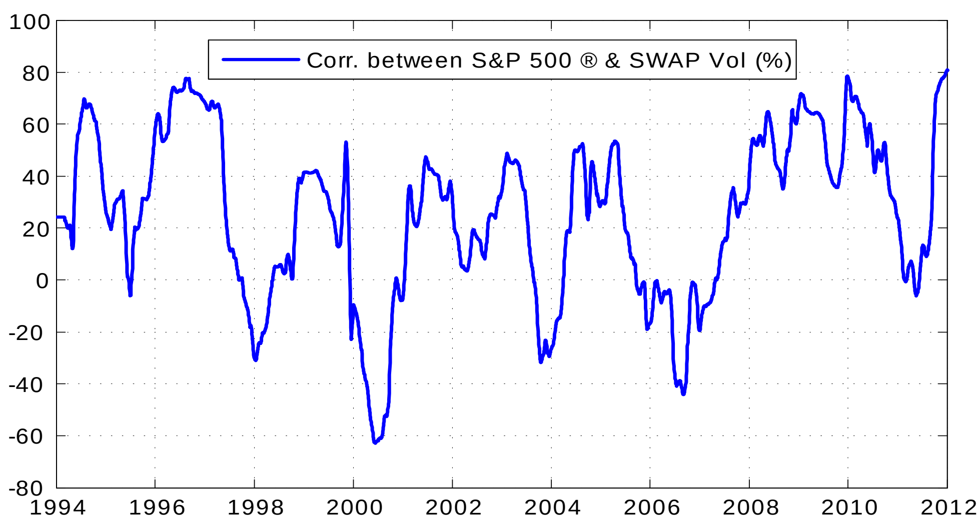

FIGURE 2. Moving average estimates of the correlation between the 20 day realized volatilities of the S&P 500® and the 10 year Swap Rate logarithmic changes. Each correlation estimate is calculated over the previous one year of data, and the sample includes daily data from January 29, 1993 to May 22, 2012, for a total of 4865 observations.

counterparts to the European-style equity options underlying equity volatility (e.g., Demeterfi, Derman, Kamal and Zou, 1999), but, also, (ii) the price of zero coupon bonds with maturities corresponding to the fixed payment days of forward swaps underlying swaptions. As such, our contract designs lead to new indexes that reflect market expectations about swap market volatility, adjusted for risk.

The most basic example of contracts we consider is structured as follows. At time t, two counterparties A and B agree that at time T, A shall pay B the realized variance of the forward swap rate from t to T, scaled by the price value of a basis point of the fixed leg of the forward swap at the n fixed payment dates $T, \cdots, T_{n-1}$ for the payment periods $T_1 - T, \cdots, T_n - T_{n-1}$ (see Figure 3).

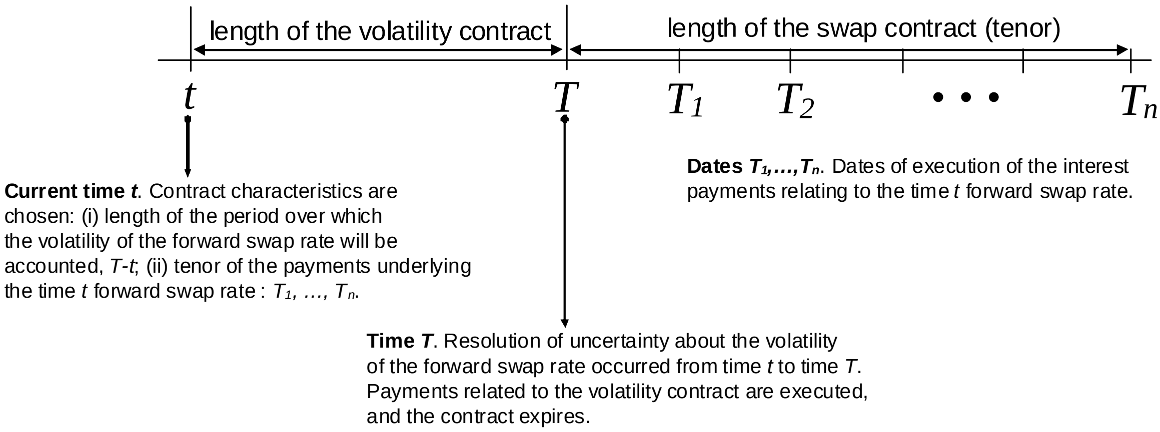

FIGURE 3. Timing of the interest rate variance contract and the execution of swap payments.

As the pricing of this and other variance swap contracts introduced below rely on bonds and swaptions, the next section provides a concise summary of risks inherent in relevant interest rate transactions. Section 3 highlights pitfalls arising from option-based trading strategies aimed to express views about developments in swap rate volatility. These pitfalls motivate the introduction of new interest rate variance contracts. Sections 4 and 5 develop the design, pricing, hedging for these new contracts: Section 4 deals with contracts relying on "percentage" volatility, i.e. the volatility of the logarithmic changes in the forward swap rate; Section 5 introduces contracts on "basis point" or "Gaussian" volatility, related to absolute changes in the forward swap rate. The contracts in Section 4 and 5 are priced in a model-free format, through "spanning arguments" such as those in Bakshi and Madan (2000) and Carr and Madan (2001), although, as noted already, the payoffs and contract designs we consider here are quite distinct since they relate to fixed income market volatility and pose new methodological issues. Our contracts and indexes aim to match the market practice of quoting swaption implied volatility both in terms of percentage and basis point implied volatility. Section 6 suggests potential trading strategies for the variance contracts introduced in Sections 4 and 5. Section 7 develops indexes of swap rate volatility, which reflect the fair value of the variance contracts of Sections 4 of 5. Section 8 explains in detail the main differences between rate variance contracts and indexes and those already in place in the equity case. Section 9 contains extensions where swap markets can experience jumps, and show the remarkable property that even in these markets, our basis point index is expressed in a model-free format. Section 10 uses sample market data to illustrate a numerical implementation of one of the interest rate swap volatility indexes. An appendix contains technical details.

2. Interest rate transactions risks

Consider a forward starting swap, i.e. a contract agreed at time t, starting at time T, and having n reset dates $T_0, \cdots, T_{n-1}$ and payment periods $T_1 - T_0, \cdots, T_n - T_{n-1}$, with $T \leq T_0$, summing up to a *tenor* period equal to $T_n - T_0$. Standard market practice sets $T = T_0$, as in Figure 3, although this practice does not play any role for all the pricing and contract designs put forward in this paper. Let $R_t(T_1, \cdots, T_n)$ be the forward swap rate prevailing at time t, i.e. the fixed rate such that the value of the forward starting swap is zero at t. In a swaption contract agreed at time t, one counterparty has the right, but not the obligation, to enter a swap contract at $T_0 < T_1$, either as a fixed rate payer or as a fixed rate receiver, with the agreed fixed rate equal to some strike K. If the swaption is exercized at T, the value of the underlying swap is, from the perspective of the fixed rate payer:

$$\mathrm{SWAP}_T(K; T_1, \cdots, T_n) = \mathrm{PVBP}_T(T_1, \cdots, T_n)\left[R_T(T_1, \cdots, T_n) - K\right], \tag{1}$$

where $\mathrm{PVBP}_T(T_1, \cdots, T_n)$ is the present value of receiving one dollar at each fixed payment date at the swaption expiry, also known as the swap's "price value of the basis point"(see, e.g., Brigo and Mercurio, 2006), i.e. the present value impact of one basis point move in the swap rate at T, $R_T(T_1, \cdots, T_n)$. Naturally, $R_\tau(T_1, \cdots, T_n)$ is both a forward and a spot swap rate when $\tau = T$. Appendix A contains a succinct summary of the evaluation framework underlying swaption contracts, as well as the mathematical definitions of both the forward swap rate, $R_t(T_1, \cdots, T_n)$, and the price value of the basis point, $\mathrm{PVBP}_t(T_1, \cdots, T_n)$, (see Eqs. (A.1) and (A.2)).

Swaptions can be priced through the Black (1976) formula once we assume the forward swap rate has deterministic volatility. The standard market practice is to use this formula to convert premiums to implied Black volatilities and vice-versa. There are two market conventions for quoting implied volatilities: one in terms of "percentage" implied volatilities, similar to equity options; and another in terms of "basis point" implied volatilities, which are percentage volatilities multiplied by the current level of the forward swap rate. This paper develops variance contracts and volatility indexes that match both strands of market practice.

An important feature of swaptions is that the underlying swap value is unknown until the swaption expiration not only because forward swap rates fluctuate but also because the PVBP of the forward swap's fixed leg also fluctuates. This PVBP is the present value at time T of each dollar arising from paying the fixed strike rate K rather than the realized swap rate $R_T(T_1, \cdots, T_n)$. It can be interpreted as a portfolio of zero coupon bonds maturing at the fixed payment dates of the swap, weighted by the lengths of the reset intervals and is determined by the yield curve prevailing at time T. Options on equities relate to a single source of risk, the stock price. Swaps, instead, are tied to two sources of risk: (i) the forward swap rate, and (ii) the swap's PVBP realized at time T. This adds complexity to defining and pricing swap market volatility.

3. Option-based volatility strategies

The standard instruments for trading interest rate swap volatility are swaption payers and receivers. Consider the following strategies:

(a) A delta-hedged swaption position, i.e. a long position in a swaption, be it payer or receiver, hedged through Black's (1976) delta.

(b) A straddle, i.e. a long position in a payer and a receiver with the same strike, maturity, and tenor.

Both strategies are analyzed in Appendix. In Appendix B, Eq. (B.9), we show the delta-hedged swaption position leads, approximately, to the following daily P&L at the swaption's expiry:

$$\mathrm{P\&L}_T \approx \frac{1}{2}\sum_{t=1}^{T}\Gamma_t^{\$}\cdot\left[\left(\sigma_t^2-\mathrm{IV}_0^2\right)\mathrm{PVBP}_T\right]+\sum_{t=1}^{T}\mathrm{Track}_t\cdot\mathrm{Vol}_t\left(\mathrm{PVBP}\right)\cdot\widetilde{\frac{\Delta R_t}{R_t}}, \tag{2}$$

where $\Gamma_t^{\$}$ is the Dollar Gamma, i.e. the swaption's Gamma times the square of the forward swap rate, σ_t is the instantaneous volatility of the forward swap rate at day t, IV_0 is the swaption implied percentage volatility at the time the strategy is first implemented, Track_t is the tracking error of the hedging strategy, $\mathrm{Vol}_t\left(\mathrm{PVBP}\right)$ is the volatility of the PVBP rate of growth at t and, finally, $\widetilde{\frac{\Delta R_t}{R_t}}$ denotes the series of shocks affecting the forward swap rate.

In Appendix B, Eq. (B.11), we show that a straddle strategy leads to a P&L similar to that in Eq. (2), with the first term being twice as that in Eq. (2), and with the straddle value replacing the tracking error in the second term:

$$\mathrm{P\&L}_T^{\text{straddle}} \approx \sum_{t=1}^{T}\Gamma_t^{\$}\cdot\left[\left(\sigma_t^2-\mathrm{IV}_0^2\right)\mathrm{PVBP}_T\right]+\sum_{t=1}^{T}\mathrm{STRADDLE}_t\cdot\mathrm{Vol}_t\left(\mathrm{PVBP}\right)\cdot\widetilde{\frac{\Delta R_t}{R_t}}, \tag{3}$$

where $\mathrm{STRADDLE}_t$ is the value of the straddle as of time t. The advantage of this strategy over the first, is that it does not rely on hedging and, hence, it is not expensive in this respect. However, it is striking to see how similar the P&L in Eqs. (2) and (3) are.

We emphasize that the P&L in Eqs. (2) and (3) are only approximations to the exact expressions given in Appendix B. For example, the exact version of the P&L in Eq. (3) has an additional term, arising because trading straddles implies a delta that is only approximately zero. It may well occur that in episodes of pronounced interest rate swap volatility, this delta drifts significantly away from zero. Naturally, not including this term in the approximations of Eqs. (2) and (3) favorably biases the assessment of option-based based strategies to trade interest rate swap volatility. This term would only add uncertainty to outcomes, on top of the uncertainties singled out below.

Eqs. (2) and (3) show that the two option-based volatility strategies lead to quite unpredictable profits that fail to isolate changes in interest rate swap volatility. There are two main issues:

(i) For both strategies, the first component of the P&L is proportional to the sum of the daily P&L, given by the "volatility view" $\sigma_t^2-\mathrm{IV}_0^2$, weighted with the Dollar Gamma, $\Gamma_t^{\$}$, and the

PVBP at the swaption expiry. This is similar, but distinct (due to the PVBP at T), from what we already know from equity option trading (see Bossu, Strasser and Guichard, 2005, and Chapter 10 in Mele, 2011). Indeed, one reason *equity* variance contracts are attractive compared to options-based strategies is that they overcome the "price dependency" the latter generates. This dependency shows up in our context as well: even when $\sigma_t^2 - \mathrm{IV}_0^2 > 0$ for most of the time, it may happen too often that bad realizations of volatility, $\sigma_t^2 - \mathrm{IV}_0^2 < 0$, occur precisely when the Dollar Gamma $\Gamma_t^{\$}$ is high. In other words, the first terms in Eq. (2) and Eq. (3) can lead to losses even if σ_t^2 is higher than IV_0^2 for most of the time.

(ii) The second terms in Eqs. (2) and (3) show that option-based P&L depend on the shocks driving the forward swap rate, $\widetilde{\frac{\Delta R_t}{R_t}}$. In fact, even if future volatility is *always* higher than the current implied volatility, $\sigma_t^2 > \mathrm{IV}_0^2$, the second terms in Eq. (2) and Eq. (3) may actually dwarf the first in the presence of adverse realizations of $\widetilde{\frac{\Delta R_t}{R_t}}$. These second terms in the P&L do not appear in option-based P&L for equity volatility. They are specific to what swaptions are: instruments that have as underlying a swap contract, whose underlying swap value's PVBP is unknown until the maturity of the swaption, as Eq. (1) shows. Uncertainty related to the PVBP, $\mathrm{Vol}_t\,(\mathrm{PVBP}) > 0$, cannot be hedged away through the previous two option-based strategies.

These two issues lead to compelling *theoretical* reasons why options might not be appropriate instruments to trade interest rate swap volatility. How reliable are option-based strategies for that purpose in practice? Figure 4 plots the P&L of two directional volatility trading strategies against the "realized variance risk-premium," defined below, and calculated using daily data from January 1998 to December 2009.

The two strategies are long positions in: (i) an at-the-money straddle; (ii) one of the interest rate variance contracts introduced in the next section. Both positions relate to contracts expiring in one and three months and tenors of five years. The realized variance risk-premium is defined as the difference between the realized variance of the forward swap rate during the relevant holding period (i.e. one or three months), and the squared implied at-the-money volatility of the swaption (at one or three months) prevailing at the beginning of the holding period. Appendix D provides all technical details regarding these calculations.

The top panel of Figure 4 depicts the two P&Ls relating to one-month trades (147 trades), and the bottom panel displays the two P&Ls relating to three-month trades (49 trades). Do these strategies deliver results consistent with directional views? In general, the P&Ls of both the straddle-based strategy and the interest rate variance contract have the same sign as the realized variance risk-premium. However, the straddle P&Ls are "dispersed," in that they do not always preserve the same sign of the realized variance risk-premium, a property displayed by the interest rate variance contract. With straddles, the P&L has the same sign as the variance risk-premium approximately 62% of the time for one-month trades, and for approximately 65% of the time for three-month trades. The correlation between the straddle P&L and the variance risk-premium is about 33% for the one-month trade, and about 28% for the three-month trade. Further simulation studies performed by Jiang (2011) under a number of alternative holding period assumptions suggest that if dynamically delta-hedged after inception, these trades lead to higher correlations—with an average correlation over all the simulation experiments equal to 63%. Instead, note the performance of the interest rate variance contract in our experiment, which is a pure play in volatility, with its P&L lining up to a straight line and a correlation with the variance risk-premium indistinguishable from 100%.

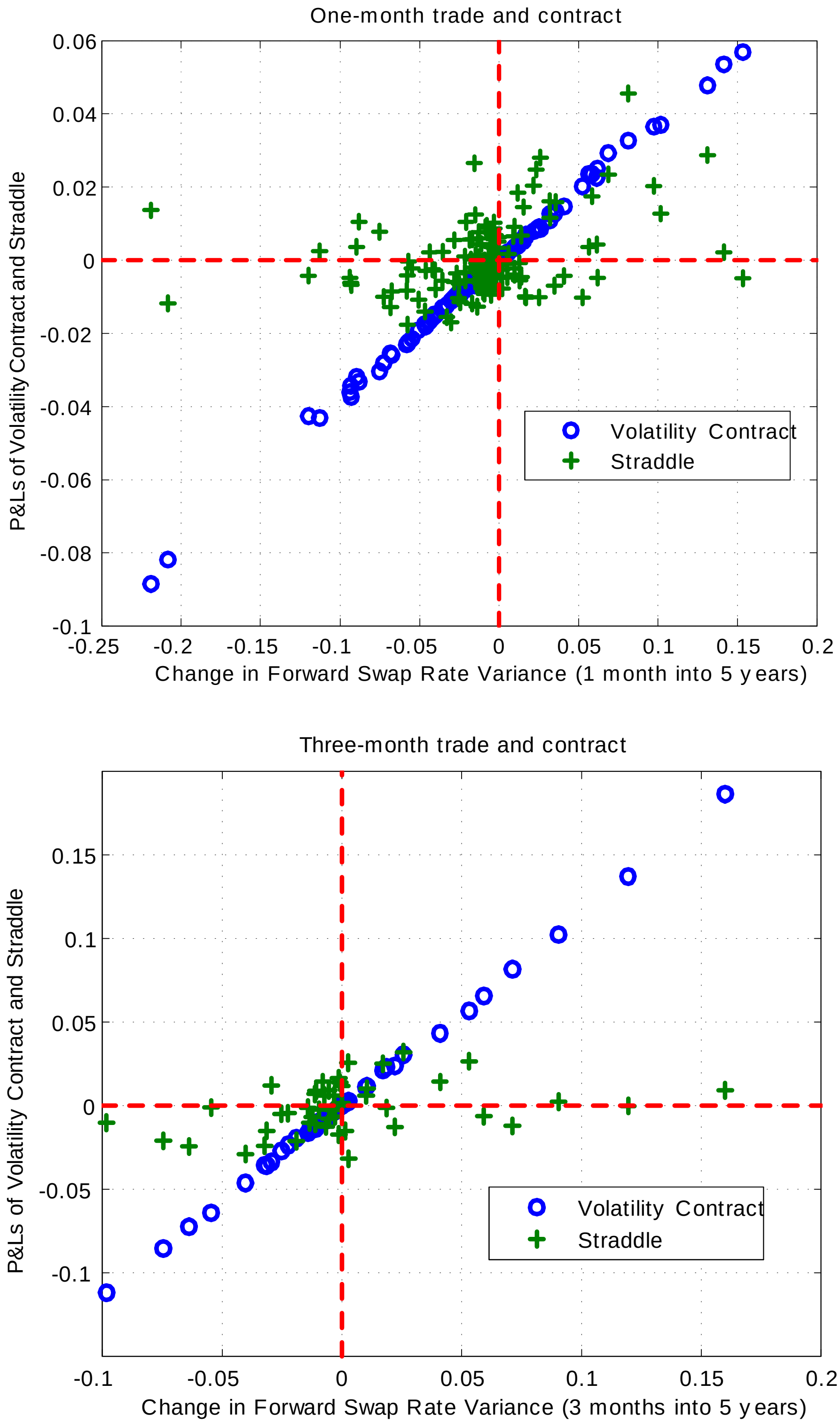

Figure 4. Empirical performance of directional volatility trades.

The reason straddles lead to more dispersed profits in the three-month trading period relates to "price dependency." For example, even if volatility is up for most of the time in a one month period, it might be down during a given week, which influences the P&L. However, straddles do not weight these ups and downs in the same way. Rather, they might attribute a large weight (given by the Dollar Gamma) to the particular week when volatility is down, as Eq. (3) makes clear. The chances these unfortunate weightings happen increase with the duration of the trade. While these facts are well-known in the case of equity volatility, the complexity of interest rate transactions makes these facts even more severe when it comes to trading interest rate swap volatility through options.

4. Interest rate swap variance contracts

We assume that the forward swap rate $R_t\left(T_1,\cdots,T_n\right)$ is a diffusion process with stochastic volatility, and consider extensions to a jump-diffusion case in Section 9. It is well-known that the forward swap rate is a martingale under the so-called *swap probability*, such that in a diffusive environment,

$$d\ln R_\tau\left(T_1,\cdots,T_n\right)=-\frac{1}{2}\left\|\sigma_\tau\left(T_1,\cdots,T_n\right)\right\|^2 d\tau+\sigma_\tau\left(T_1,\cdots,T_n\right)\cdot dW_\tau^*,\quad \tau\in[t,T], \tag{4}$$

where W_t^* is a multidimensional Wiener process under the *swap probability*, and $\sigma_t\left(T_1,\cdots,T_n\right)$ is adapted to W_t^*. Accordingly, define $V_n(t,T)$, as the realized variance of the forward swap rate logarithmic changes in the time interval $[t,T]$, for tenor length T_n-T:

$$V_n(t,T)\equiv\int_t^T\left\|\sigma_\tau\left(T_1,\cdots,T_n\right)\right\|^2 d\tau. \tag{5}$$

Appendix A summarizes additional notation and basic assumptions and facts about forward starting swaps, swap rates, the swap probability, and the pricing of swaptions.

This section designs contracts to trade the forward swap rate volatility, $V_n(t,T)$, that overcome the issues inherent in option trading as observed in Section 3. The contracts are priced in a model-free fashion, and lead to model-free indexes of expected volatility over a reference period $[t,T]$—introduced in Section 7. This section considers contracts referencing *percentage*, or *logarithmic* variance, and Section 5 develops their *basis point* counterparts.

4.1. Contracts

We consider three contracts: a forward agreement, which requires an initial payment, and two variance swaps, which are settled at the expiration. These contracts are all equally useful for the purpose of implementing views about swap market volatility developments, as explained in Section 6.

We begin with the forward agreement. Our first remark is that uncertainty about swap markets relates to the volatility of the forward swap rate, rescaled by the PVBP, as Eq. (1) reveals, with the complication that the PVBP is unknown until T. Consider the definition of the realized variance of the forward swap rate, $V_n(t,T)$ in Eq. (5), and the following forward agreement:

> **Definition I (Interest Rate Variance Forward Agreement)**. At time t, counterparty A promises to pay counterparty B the product of forward swap rate variance realized over the interval $[t, T]$ times the swap's PVBP prevailing at time T, i.e. the value $V_n(t,T) \times \text{PVBP}_T(T_1, \cdots, T_n)$. The price counterparty B shall pay counterparty A for this agreement at time t is called the Interest Rate Variance (IRV) Forward rate, and is denoted as $\mathbb{F}_{\text{var},n}(t,T)$.

The PVBP is the price impact of a one basis point move in the swap rate over the tenor at time T, which is unknown before time T, as further discussed after Definition III. Rescaling by the forward PVBP is mathematically unavoidable when the objective is to price volatility in a model-free fashion. Our goal is to simultaneously price interest rate swap volatility, swaps, swaptions and pure discount bonds in such a way that the price of volatility conveys all the information carried by all the traded swaptions and bonds. Eq. (1) suggests that the uncertainty related to the interest rate swaps underlying the swaptions is tied to both the realized variance, $V_n(t,T)$, and the forward PVBP. To insulate and price volatility, the payoff of the variance contract needs to be rescaled by $\text{PVBP}_T(T_1, \cdots, T_n)$, just as the value of the swap does in Eq. (1). Note that the term $\text{PVBP}_T(T_1, \cdots, T_n)$ also appears in the P&L of the swaption-based volatility strategies, as shown by Eqs. (2) and (3); the important difference is that the variance contract underlying Definition I isolates volatility from Dollar Gamma.[1] Finally, from a practical perspective, such a rescaling does not affect the ability of the contract to convey views about future developments of $V_n(t,T)$: as noted, the correlation between the variance risk-premium and the P&L of IR-variance contracts is nearly 100% in the empirical experiment in Figure 4.

In fact, the P&L in Figure 4 refer to a variance contract cast in a "swap format," whereby counterparty A agrees to pay counterparty B the following difference at time T:

$$\text{Var-Swap}_n(t,T) \equiv V_n(t,T) \times \text{PVBP}_T(T_1, \cdots, T_n) - \mathbb{P}_{\text{var},n}(t,T), \tag{6}$$

where $\mathbb{P}_{\text{var},n}(t,T)$ is a fixed variance swap rate, determined at time t, chosen in a way to lead to the following zero value condition at time t:

> **Definition II (IRV Swap Rate)**. The IRV swap rate is the fixed variance swap rate $\mathbb{P}_{\text{var},n}(t,T)$, which makes the current value of $\text{Var-Swap}_n(t,T)$ in Eq. (6) equal to zero.

Note that the variance contract arising from Definition II is a forward, not really a swap. However, we utilize a terminology similar to that already in place for equity variance contracts, and simply refer to the previous contract as a "swap."

Finally, we consider a second definition of the IRV swap rate, which will lead to a definition of the index of interest rate swap volatility in Section 7. Consider the following payoff:

$$\text{Var-Swap}_n^*(t,T) \equiv \left[V_n(t,T) - \mathbb{P}^*_{\text{var},n}(t,T)\right] \times \text{PVBP}_T(T_1, \cdots, T_n), \tag{7}$$

where $\mathbb{P}^*_{\text{var},n}(t,T)$, a fixed variance swap rate determined at t, is set so as to lead to a zero value condition at time t:

[1] If the risk-neutral expectation of the short-term rate is insensitive to changes in the short-term swap rate volatility, the PBVT at time T is increasing in the short-term swap rate volatility (see Mele, 2003). In this case, the IRV forward agreement is a device to lock-in both volatility of the forward swap rate *from t to T, and* short-term swap rate volatility *at T*.

Definition III (Standardized IRV Swap Rate). The Standardized IRV swap rate is the fixed variance swap rate $\mathbb{P}^{*}_{\mathrm{var},n}(t,T)$ that makes the current value of Var-Swap$^{*}_{n}(t,T)$ in Eq. (7) equal to zero.

The payoff in Eq. (7) carries an intuitive meaning. It is the product of two terms: (i) the difference between the realized variance and a fair strike and (ii) the PVBP at time T. This payoff is similar to that of standard equity variance contracts: the holder of the contract would receive $\mathrm{PVBP}_T(T_1,\cdots,T_n)$ dollars for every point by which the realized variance exceeds the variance strike price. The added complication of the IRV contract design is that $\mathrm{PVBP}_T(T_1,\cdots,T_n)$ is unknown at time t.

4.2. Pricing

In the absence of arbitrage and market frictions, the price of the IRV forward agreement to be paid at time t, $\mathbb{F}_{\mathrm{var},n}(t,T)$ in Definition I, is expressed in a model-free format as a combination of the prices of at-the-money and out-of-the-money swaptions. One approximation to $\mathbb{F}_{\mathrm{var},n}(t,T)$ based on a finite number of swaptions, is, keeping the same notation:

$$\boxed{\mathbb{F}_{\mathrm{var},n}(t,T)=2\left[\sum_{i:K_i<R_t}\frac{\mathrm{SWPN}^{\mathrm{R}}_t(K_i,T;T_n)}{K_i^2}\Delta K_i+\sum_{i:K_i\geq R_t}\frac{\mathrm{SWPN}^{\mathrm{P}}_t(K_i,T;T_n)}{K_i^2}\Delta K_i\right]}\tag{8}$$

where $\mathrm{SWPN}^{\mathrm{R}}_t(K_i,T;T_n)$ (resp., $\mathrm{SWPN}^{\mathrm{P}}_t(K_i,T;T_n)$) is the price of a swaption receiver (resp., payer), struck at K_i, expiring at T and with tenor extending up to time T_n, and $\Delta K_i=\frac{1}{2}(K_{i+1}-K_{i-1})$ for $1\leq i<M$, $\Delta K_0=(K_1-K_0)$, $\Delta K_M=(K_M-K_{M-1})$, where K_0 and K_M are the lowest and the highest available strikes traded in the market, and $M+1$ is the total number of traded swaptions expiring at time T and with tenor extending up to time T_n. Appendix C provides the theoretical framework underlying Eq. (8). Appendix E provides an expression for the local volatility surface, which we can use to "interpolate" the missing swaptions from those that are traded, so as to "fill in" Eq. (8).

In Appendix C, we show that the swap rate in Definition II is given by:

$$\boxed{\mathbb{P}_{\mathrm{var},n}(t,T)=\frac{\mathbb{F}_{\mathrm{var},n}(t,T)}{P_t(T)}}\tag{9}$$

where $P_t(T)$ is the price of a zero coupon bond maturing at T and time-to-maturity $T-t$. Finally, the standardized swap rate in Definition III is:

$$\boxed{\mathbb{P}^{*}_{\mathrm{var},n}(t,T)=\frac{\mathbb{F}_{\mathrm{var},n}(t,T)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}}\tag{10}$$

4.3. Hedging

The interest rate variance contracts can be hedged using swaptions underlying the evaluation framework of Section 4.2. An IRV market maker will be concerned with the replication of the payoffs needed to hedge the contracts. In the exposition below, we take the perspective of a provider of insurance against volatility who sells any of the IRV contracts underlying Definitions I through III of Section 4.1.

The IRV forward contract in Definition I can be hedged through the positions in rows (i) and (ii) of Tables I and II, which we now discuss. Next, consider the contracts relating to the IRV swap rate of Definition II, $\mathbb{P}_{\text{var},n}(t,T)$. We aim to replicate the payoff in Eq. (6). As shown in Appendix C, we hedge against this payoff through two, identical, zero cost portfolios, constructed with the following positions:

(i) A dynamic position in a self-financed portfolio of zero coupon bonds aiming to replicate the cumulative increments of the forward swap rate over $[t,T]$, which turns out to equal $\int_t^T \frac{dR_s(T_1,\cdots,T_n)}{R_s(T_1,\cdots,T_n)} = \frac{1}{2}V_n(t,T) + \ln\frac{R_T(T_1,\cdots,T_n)}{R_t(T_1,\cdots,T_n)}$, rescaled by the PVBP at time T.

(ii) Static positions in swaps starting at time T and OTM swaptions expiring at time T, aiming to replicate the payoff of the interest rate counterpart to the log-contract (Neuberger, 1994) for equities, rescaled by the PVBP at T. The positions are as follows:

  (ii.1) Short $1/R_t(T_1,\cdots,T_n)$ units of a forward starting (at T) fixed interest payer swap struck at the current forward swap rate, $R_t(T_1,\cdots,T_n)$.

  (ii.2) Long out-of-the-money swaptions, each of them carrying a weight equal to $\frac{\Delta K_i}{K_i^2}$, where ΔK_i and K_i are as in Eq. (8).

(iii) A static borrowing position aimed to finance the out-of-the-money swaption positions in (ii.2).

The details for the self-financed portfolio in (i) are given in Appendix C. Table I summarizes the costs of each of these trades at t, as well as the payoffs at T, which sum up to be the same as the payoff in Eq. (6), as required to hedge the contract.

Table I

Replication of the contract relying on the IRV swap rate of Definition II. ZCB and OTM stand for zero coupon bonds and out-of-the-money, respectively, and $R_\tau \equiv R_\tau(T_1,\cdots,T_n)$, $\text{PVBP}_T \equiv \text{PVBP}_T(T_1,\cdots,T_n)$.

Portfolio	Value at t	Value at T
(i) long self-financed portfolio of ZCB	0	$\left(V_n(t,T) + 2\ln\frac{R_T}{R_t}\right) \times \text{PVBP}_T$
(ii) short swaps and long OTM swaptions	$-\mathbb{F}_{\text{var},n}(t,T)$	$-2\ln\frac{R_T}{R_t} \times \text{PVBP}_T$
(iii) borrow $\mathbb{F}_{\text{var},n}(t,T)$	$+\mathbb{F}_{\text{var},n}(t,T)$	$-\frac{\mathbb{F}_{\text{var},n}(t,T)}{P_t(T)} = -\mathbb{P}_{\text{var},n}(t,T)$
Net cash flows	0	$V_n(t,T) \times \text{PVBP}_T - \mathbb{P}_{\text{var},n}(t,T)$

Finally, to hedge against the Standardized contract of Definition III, we need to replicate the payoff in Eq. (7). We create a portfolio, which is the same as that in Table I, except now that row (iii) refers to a borrowing position in a basket of zero coupon bonds with value equal to $\text{PVBP}_t(T_1,\cdots,T_n)$ and notional of $\mathbb{P}^*_{\text{var},n}(t,T)$, as in Table II. By Eq. (10), the borrowing position needed to finance the out-of-the-money swaptions in row (ii), is just $\mathbb{F}_{\text{var},n}(t,T) = \mathbb{P}^*_{\text{var},n}(t,T) \times \text{PVBP}_t(T_1,\cdots,T_n)$. Come time T, it will be closed for a value of $-\mathbb{P}^*_{\text{var},n}(t,T) \times \text{PVBP}_T(T_1,\cdots,T_n)$. The net cash flows are precisely those of the contract with the Standardized IRV swap rate.

Table II

Replication of the contract relying on the Standardized IRV swap rate of Definition III. ZCB and OTM stand for zero coupon bonds and out-of-the-money, respectively, and $R_\tau \equiv R_\tau(T_1,\cdots,T_n)$, $\text{PVBP}_\tau \equiv \text{PVBP}_\tau(T_1,\cdots,T_n)$.

Portfolio	Value at t	Value at T
(i) long self-financed portfolio of ZCB	0	$\left(V_n(t,T)+2\ln\frac{R_T}{R_t}\right)\times\text{PVBP}_T$
(ii) short swaps and long OTM swaptions	$-\mathbb{F}_{\text{var},n}(t,T)$	$-2\ln\frac{R_T}{R_t}\times\text{PVBP}_T$
(iii) borrow basket of ZCB for $\mathbb{P}^*_{\text{var},n}(t,T)\times\text{PVBP}_t$	$+\mathbb{F}_{\text{var},n}(t,T)$	$-\mathbb{P}^*_{\text{var},n}(t,T)\times\text{PVBP}_T$
Net cash flows	0	$\left[V_n(t,T)-\mathbb{P}^*_{\text{var},n}(t,T)\right]\times\text{PVBP}_T$

The replication arguments in this section are alternative means to determine the no-arbitrage value of the IRV forward agreement of Definition I, and the IRV swap rates in Definitions II and III, $\mathbb{P}_{\text{var},n}(t,T)$ and $\mathbb{P}^*_{\text{var},n}(t,T)$. Suppose, for example, that the market value of the Standardized IRV swap rate, $\mathbb{P}^*_{\text{var},n}(t,T)_\$$ say, is higher than the no-arbitrage value $\mathbb{P}^*_{\text{var},n}(t,T)$ in Eq. (10). Then, one could short the standardized contract underlying Definition III and, at the same time, synthesize it through the portfolio in Table II. This positioning costs zero at time t and yields a sure positive profit at time T equal to $\left[\mathbb{P}^*_{\text{var},n}(t,T)_\$-\mathbb{P}^*_{\text{var},n}(t,T)\right]\times\text{PVBP}_T$. To rule out arbitrage, then, we need to have $\mathbb{P}^*_{\text{var},n}(t,T)_\$=\mathbb{P}^*_{\text{var},n}(t,T)$.

5. Basis point volatility

5.1. Contracts and evaluation

We can price and hedge volatility based on *arithmetic*, or *basis point* (BP henceforth), changes of the forward swap rate in Eq. (4),

$$dR_\tau(T_1,\cdots,T_n)=R_\tau(T_1,\cdots,T_n)\,\sigma_\tau(T_1,\cdots,T_n)\,dW^*_\tau,\quad \tau\in[t,T]. \tag{11}$$

Accordingly, denote the BP realized variance with $V_n^{\text{BP}}(t,T)$,

$$V_n^{\text{BP}}(t,T)\equiv\int_t^T R_\tau^2(T_1,\cdots,T_n)\left\|\sigma_\tau(T_1,\cdots,T_n)\right\|^2 d\tau \tag{12}$$

We can then restate the interest rate variance contracts in Section 4.1 in terms of BP variance and, for reasons explained below, we refer to the ensuing contracts as “Gaussian” contracts, defined as follows:

Definition IV (Gaussian Contracts). Consider the following contracts and swap rates replacing those in Definitions I, II, and III:

(a) **(BP-IRV Forward Agreement)**. At time t, counterparty A promises to pay counterparty B the product of the forward swap rate BP variance realized over the interval $[t,T]$ times the swap's PVBP that will prevail at time T, i.e. the value $V_n^{\text{BP}}(t,T)\times\text{PVBP}_T(T_1,\cdots,T_n)$. The price counterparty B shall pay counterparty A for this agreement at time t is called BP-IRV Forward rate, and is denoted as $\mathbb{F}^{\text{BP}}_{\text{var},n}(t,T)$.

(b) **(BP-IRV Swap Rate)**. The BP-IRV swap rate is the fixed variance swap rate $\mathbb{P}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)$, which makes the current value of $V^{\mathrm{BP}}_n(t,T) \times \mathrm{PVBP}_T(T_1,\cdots,T_n) - \mathbb{P}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)$ equal to zero.

(c) **(Standardized BP-IRV Swap Rate)**. The Standardized BP-IRV swap rate is the fixed variance swap rate $\mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T)$, which makes the current value of $\left[V^{\mathrm{BP}}_n(t,T) - \mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T)\right] \times \mathrm{PVBP}_T(T_1,\cdots,T_n)$ equal to zero.

The reason we refer to these contracts as "Gaussian" relates to the fact that a benchmark case is one where the realized BP variance of the forward swap rate in Eq. (11) is constant and equal to σ_N,

$$dR_\tau(T_1,\cdots,T_n) = \sigma_N dW^*_\tau, \quad \tau \in [t,T], \tag{13}$$

such that the forward swap rate has a Gaussian distribution. We shall return to this abstract assumption of a constant BP variance below to develop intuition about the pricing results relating to the general stochastic BP variance case as defined in Eq. (12).

In Appendix G, we show that in the general case of Eqs. (11) and (12), the price of the BP-IRV forward contract is approximated by:

$$\boxed{\mathbb{F}^{\mathrm{BP}}_{\mathrm{var},n}(t,T) \equiv 2\left[\sum_{i:K_i<R_t} \mathrm{SWPN}^{\mathrm{R}}_t(K_i,T;T_n)\,\Delta K_i + \sum_{i:K_i\geq R_t} \mathrm{SWPN}^{\mathrm{P}}_t(K_i,T;T_n)\,\Delta K_i\right]} \tag{14}$$

where the strike differences, ΔK_i, are as in Eq. (8). Given the BP-IRV forward, the BP-IRV swap rates in Definition IV-(a) and Definition IV-(b) are:

$$\boxed{\mathbb{P}^{\mathrm{BP}}_{\mathrm{var},n}(t,T) = \frac{\mathbb{F}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)}{P_t(T)} \qquad \text{and} \qquad \mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T) = \frac{\mathbb{F}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}} \tag{15}$$

The intuition behind the forward price in Eq. (14) is that the instantaneous BP-variance is simply the instantaneous variance of the logarithmic changes of the forward swap rate rescaled by the forward swap rate, as in Eq. (11), and squared. This property translates into an analogous property for the price of the forward variance agreement: comparing Eq. (8) with Eq. (14) reveals that for the BP agreement, each swaption price i carries the same weight as that for the agreement in Definition I, $\frac{\Delta K_i}{K_i^2}$, rescaled by the squared strike K_i^2, i.e. $\frac{\Delta K_i}{K_i^2} \times K_i^2$.

In their derivation of the fair value of equity variance swaps, Demeterfi, Derman, Kamal and Zou (1999) develop an intuitive approach relying on the Black and Scholes (1973) market, and explain that a portfolio of options has a vega that is insensitive to changes in the stock price only when the options are weighted inversely proportional to the square of the strike, which also relates to the fair value of our forward variance agreement in Eq. (8). We develop a similar approach to gain intuition about the reasons underlying the uniform swaptions weightings in Eq. (14).

Assume a Gaussian market, i.e. one where the forward swap rate is as in Eq. (13). This assumption is the obvious counterpart to that of a constant percentage volatility underlying the standard Black-Scholes market for equity options. Denote with $\mathcal{O}_t(R,K,T,\sigma_N;T_n)$ the price of a swaption (be it a receiver or a payer) at time t when the forward swap rate is R. We create a portfolio with a continuum of swaptions having the same maturity and tenor, and denote with $\omega(K)$ the portfolio weigthings, assumed to be independent of R, and such that the value of the portfolio is:

$$\pi_t(R_t(T_1,\cdots,T_n),\sigma_N) \equiv \int \omega(K)\,\mathcal{O}_t(R_t(T_1,\cdots,T_n),K,T,\sigma_N;T_n)\,dK.$$

We require that the vega of the portfolio, defined as $\nu_t(R,\sigma) \equiv \frac{\partial \pi_t(R,\sigma)}{\partial \sigma}$, be insensitive to changes in the forward swap rate,

$$\frac{\partial \nu_t(R,\sigma)}{\partial R} = 0. \tag{16}$$

In Appendix G, we show that the vega of the portfolio does not respond to R if and only if the weightings, $\omega(K)$, are independent of K,

$$\text{Eq. (16) holds true} \Longleftrightarrow \omega(K) = \text{const.} \tag{17}$$

That is, a portfolio of swaptions aiming to track BP variance, thereby being immuned to changes in the underlying forward swap rate, is one where the swaptions are equally weighted.

5.2. Hedging

Replicating BP-denominated contracts requires positioning in a way quite different from replicating percentage volatility as explained in Section 4.3. As noted in the previous section, the hedging arguments summarized in Tables I and II rely on the replication of *log-contracts*, i.e. contracts delivering $\ln(\frac{R_T}{R_t})$, rescaled by the PVBP. Instead, to hedge BP contracts, we need to rely on *quadratic contracts*, i.e. those delivering $R_T^2 - R_t^2$, rescaled by the PVBP, as we now explain.

Consider, first, the payoff of the BP-IRV forward contract in Definition IV-(a), which can be replicated through the portfolios in rows (i) and (ii) of Tables III and IV, as further discussed below. To replicate the contract for the BP-IRV swap rate of Definition IV-(b), we create portfolios comprising the following positions:

(i) A dynamic, short position in two identical, self-financed portfolios of zero coupon bonds that replicate the increments of the forward swap rate over $[t,T]$, weighted by the forward swap rate, which turns out to equal $-\int_t^T R_s(T_1,\cdots,T_n)\,dR_s(T_1,\cdots,T_n) = \frac{1}{2}\left[V_n^{\mathrm{BP}}(t,T) - (R_T^2(T_1,\cdots,T_n) - R_t^2(T_1,\cdots,T_n))\right]$, rescaled by the PVBP at time T.

(ii) Static positions in swaps starting at time T and OTM swaptions expiring at time T, aiming to replicate the payoff of a "quadratic contract," rescaled by the PVBP at T. The positions are as follows:

(ii.1) Long $2R_t(T_1,\cdots,T_n)$ units of a forward starting (at T) fixed interest payer swap struck at the current forward swap rate, $R_t(T_1,\cdots,T_n)$.

(ii.2) Long out-of-the-money swaptions, each of them carrying a weight equal to $2\Delta K_i$, where ΔK_i and K_i are as in Eq. (8).

(iii) A static borrowing position aimed to finance the out-of-the-money swaption positions in (ii.2).

Costs and payoffs of these portfolios are summarized in Table III. The result is a perfect replication of the payoff relating to the contract of Definition IV-(b).

Table III

Replication of the payoff relating to the BP-IRV contract of Definition IV-(b). ZCB and OTM stand for zero coupon bonds and out-of-the-money, respectively, and $R_\tau \equiv R_\tau(T_1, \cdots, T_n)$, $\text{PVBP}_T \equiv \text{PVBP}_T(T_1, \cdots, T_n)$.

Portfolio	Value at t	Value at T
(i) short self-financed portfolio of ZCB	0	$\left[V_n^{\text{BP}}(t,T) - \left(R_T^2 - R_t^2\right)\right] \times \text{PVBP}_T$
(ii) long swaps and long OTM swaptions	$-\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$	$\left(R_T^2 - R_t^2\right) \times \text{PVBP}_T$
(iii) borrow $\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$	$+\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$	$-\frac{\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)}{P_t(T)} = -\mathbb{P}_{\text{var},n}^{\text{BP}}(t,T)$
Net cash flows	0	$V_n^{\text{BP}}(t,T) \times \text{PVBP}_T - \mathbb{P}_{\text{var},n}^{\text{BP}}(t,T)$

Finally, Table IV is obtained by modifying the arguments leading to Table III, in the same way as we did to obtain Table II in Section 4.3.

Table IV

Replication of the payoff relating to the Standardized BP-IRV contract of Definition IV-(c). ZCB and OTM stand for zero coupon bonds and out-of-the-money, respectively, and $R_\tau \equiv R_\tau(T_1, \cdots, T_n)$, $\text{PVBP}_\tau \equiv \text{PVBP}_\tau(T_1, \cdots, T_n)$.

Portfolio	Value at t	Value at T
(i) short self-financed portfolio of ZCB	0	$\left[V_n^{\text{BP}}(t,T) - \left(R_T^2 - R_t^2\right)\right] \times \text{PVBP}_T$
(ii) long swaps and long OTM swaptions	$-\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$	$\left(R_T^2 - R_t^2\right) \times \text{PVBP}_T$
(iii) borrow basket of ZCB for $\mathbb{P}_{\text{var},n}^{*\text{BP}}(t,T) \times \text{PVBP}_t$	$+\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$	$-\mathbb{P}_{\text{var},n}^{*\text{BP}}(t,T) \times \text{PVBP}_T$
Net cash flows	0	$\left[V_n^{\text{BP}}(t,T) - \mathbb{P}_{\text{var},n}^{*\text{BP}}(t,T)\right] \times \text{PVBP}_T$

5.3. *Links to constant maturity swaps*

We show that a constant maturity swap (CMS) can be represented as a basket of the BP-IRV forwards of Definition IV-(a), suggesting interest rate variance swaps can be used to hedge against CMS. The technical reason IRV forwards link to CMS is due to a classical "convexity adjustment" of the CMS, arising because the price of a CMS relates to the "wrong expectation" of future payoffs, one under which the forward swap rate is not a martingale.

In a standard constant maturity swap, a party pays a counterparty the spot swap rate with a fixed tenor over a sequence of dates, with legs set in advance, i.e. $R_{T_0+j\kappa}(T_1 + j\kappa, \cdots, T_n + j\kappa)$ at times $T_0 + (j+1)\kappa$, $j = 0, \cdots, N-1$, where κ is a fixed constant, for example $\frac{1}{2}$, and N is the number of payment dates. In Appendix G, we show that the price of a CMS is approximately equal to:

$$\begin{aligned}\text{CMS}_N(t) \equiv &\sum_{j=0}^{N-1} P_t(\tau_j + k)\, R_t(T_1 + j\kappa, \cdots, T_n + j\kappa) \\ &+ \sum_{j=0}^{N-1} G'\left(R_t(T_1 + j\kappa, \cdots, T_n + j\kappa)\right) \mathbb{F}_{\text{var},n}^{\text{BP}}(t, T_0 + j\kappa), \qquad (18)\end{aligned}$$

where $\mathbb{F}_{\text{var},n}^{\text{BP}}(t,T)$ is the price of the BP-IRV forward contract in Definition IV-(a), as given by Eq. (14), and $G(\cdot)$ is a function given in the Appendix (see Eq. (G.8)).

That the price of a CMS relates to the entire swaption skew is known at least since Hagan (2003) and Mercurio and Pallavicini (2006), as further discussed in Appendix G. As is also clear from Eq. (18), CMS can be hedged through a basket of IRV forwards in BP expiring at the points preceding the CMS payments.

6. Marking to market and trading strategies

This section provides expressions for marking to market updates of the IRV contracts of Sections 4 and 5 (see Section 6.1). These expressions also help assess the value of trading strategies based on these contracts, as explained in Section 6.2.

6.1. *Marks to market of IRV swaps*

In Appendix C, we show that the value at any time $\tau \in (t, T)$, of the variance contract struck at time t at the IRV swap rate $\mathbb{P}_{\text{var},n}(t,T)$ in Eq. (9) is:

$$\text{M-Var}_n(t,\tau,T) \equiv V_n(t,\tau) \times \text{PVBP}_\tau(T_1,\cdots,T_n) - P_\tau(T)\left[\mathbb{P}_{\text{var},n}(t,T) - \mathbb{P}_{\text{var},n}(\tau,T)\right], \quad (19)$$

and the value of the contract struck at the Standardized IRV swap rate $\mathbb{P}^*_{\text{var},n}(t,T)$ in Eq. (10) is:

$$\text{M-Var}^*_n(t,\tau,T) \equiv \text{PVBP}_\tau(T_1,\cdots,T_n)\left[V_n(t,\tau) - \left(\mathbb{P}^*_{\text{var},n}(t,T) - \mathbb{P}^*_{\text{var},n}(\tau,T)\right)\right]. \quad (20)$$

It is immediate to see that the marking to market updates for Basis Point contracts are the same as those in Eqs. (19) and (20)—differing only by a mere change in notation. In the next section, we utilize these marking to market expressions to calculate the value of trading strategies.

6.2. *Trading strategies*

We consider examples of trading strategies relying on the IRV contracts of Sections 4 and 5. We analyze strategies relying on views about developments of: (i) spot IRV swap rates relative to current (in Section 6.2.1), and (ii) the value of spot IRV swap rates relative to forward IRV swap rates—i.e. those implied by the current term-structure of spot IRV swap rates (in Section 6.2.2). The strategies we consider apply to both percentage and BP contracts, although to simplify the presentation, we only illustrate the case applying to percentage contracts.

6.2.1. *Spot trading*

6.2.1.1. *Trading through IRV swap rates*

First, we suggest how to express views on IRV contracts that have different maturities and tenors. Let $m = \frac{1}{12}$, and consider, for example, Figure 5, which illustrates the timing of two contracts: (i) a contract expiring in 3 months, relating to the volatility of $3m$ into $\hat{T}_5 - 3m$ forward swap rates; and (ii) a contract expiring in 9 months, relating to the volatility of $9m$ into $T_5 - 9m$ forward swap rates. Time units are expressed in years, and we set $t = 0$. We assume the two contracts have the same tenor, $\hat{T}_5 - 3m = T_5 - 9m$.

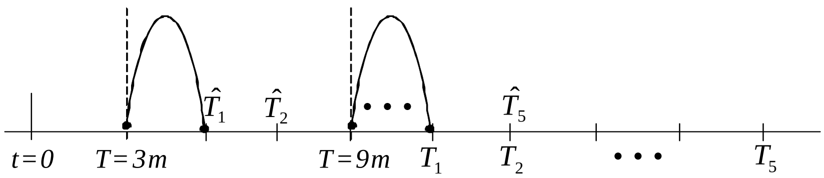

FIGURE 5. Rolling tenor trade.

Let $\mathbb{P}_{\text{var},\hat{n}}(t,T)$ and $\mathbb{P}_{\text{var},n}(t,T)$ be the IRV swap rates of these two contracts, and suppose we have the view that at some point $\tau \leq 3m$,

$$\mathbb{P}_{\text{var},\hat{n}}(\tau,3m) > \mathbb{P}_{\text{var},\hat{n}}(0,3m) \quad \text{and} \quad \mathbb{P}_{\text{var},n}(\tau,9m) < \mathbb{P}_{\text{var},n}(0,9m). \tag{21}$$

A trading strategy consistent with these views might be as follows: (i) go long the $3m$ contract, (ii) short α units of the $9m$ contract, (iii) short the IRV forward contract in Definition I for the $3m$ maturity, with contract length equal to τ, (iv) go long α units of the IRV forward contract for the $9m$ maturity, with contract length equal to τ. The value of this strategy at time τ is, by marking-to-market according to Eq. (19):

$$\begin{aligned}
&\text{M-Var}_{\hat{n}}(0,\tau,3m) - \alpha\cdot\text{M-Var}_n(0,\tau,9m)\\
&-V_{\hat{n}}(0,\tau)\times\text{PVBP}_\tau(\hat{T}_1,\cdots,\hat{T}_n) + \alpha\cdot V_n(0,\tau)\times\text{PVBP}_\tau(T_1,\cdots,T_n)\\
&= P_\tau(3m)\left[\mathbb{P}_{\text{var},\hat{n}}(\tau,3m) - \mathbb{P}_{\text{var},\hat{n}}(0,3m)\right] + \alpha\cdot P_\tau(9m)\left[\mathbb{P}_{\text{var},n}(0,9m) - \mathbb{P}_{\text{var},n}(\tau,9m)\right]\\
&> 0.
\end{aligned}$$

Its cost at $t=0$ is $-\mathbb{F}_{\text{var},\hat{n}}(0,3m) + \alpha\cdot\mathbb{F}_{\text{var},n}(0,9m)$. We choose,

$$\alpha = \frac{\mathbb{F}_{\text{var},\hat{n}}(0,3m)}{\mathbb{F}_{\text{var},n}(0,9m)}, \tag{22}$$

to make the portfolio worthless at $t=0$.

The technical reason to enter into the IRV forward contract is that the time τ value of the variance contracts (i) and (ii) entails, by Eq. (19), cumulative realized variance exposures equal to $V_{\hat{n}}(0,\tau)\times\text{PVBP}_\tau(\hat{T}_1,\cdots,\hat{T}_n)$ and $-V_n(0,\tau)\times\text{PVBP}_\tau(T_1,\cdots,T_n)$, which cannot offset, as they refer to two distinct rates: the $3m$ into $\hat{T}_5 - 3m$ forward swap rate and the $9m$ into $T_5 - 9m$ forward swap rate. The device to short α units of the $9m$ contract serves the purpose of offsetting these two exposures at time τ, and the expression for α in Eq. (22) allows us to create a zero-cost portfolio at $t=0$.

Next, we develop an example of a trading strategy relying on contracts with different maturities but the same forward swap rates. Assume that at time $t\equiv 0$, the term structure of the IRV swap rate $\mathbb{P}_{\text{var},n}(0,T)$ is increasing in T for some fixed tenor, but that a view is held that this term structure is about to flatten soon. For example, we might expect that at some point in time τ,

$$\mathbb{P}_{\text{var},n}(\tau,3m) > \mathbb{P}_{\text{var},n}(0,3m) \quad \text{and} \quad \mathbb{P}_{\text{var},n}(\tau,9m) < \mathbb{P}_{\text{var},n}(0,9m), \tag{23}$$

where $\tau \leq 3m$.

To express this view, one could go long a 3 month IRV swap struck at $\mathbb{P}_{\text{var},n}(0,3m)$ and shorts a 9 month IRV swap struck at $\mathbb{P}_{\text{var},n}(0,9m)$. Note that this trading strategy relies on

IRV swaps for forward swap rates that are 9 months into $T_n - 9m$ years, at time $t = 0$, as Figure 6 illustrates.

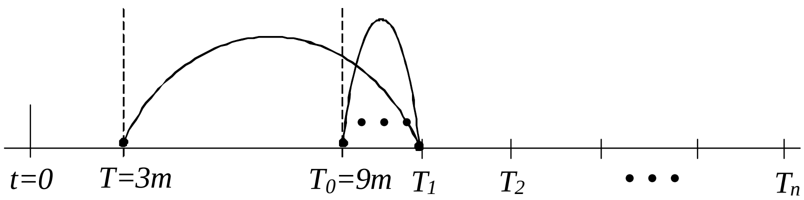

FIGURE 6. Fixed tenor trade.

The payoffs to these two IRV swaps are as follows:

(i) The payoff of the 3 month IRV swap at $T = 3m$ is $V_n(0, 3m) \times \text{PVBP}_{3m}(T_1, \cdots, T_n)$, where the $\text{PVBP}_{3m}(T_1, \cdots, T_n)$ relates to swaps payments starting at time T_1.

(ii) The payoff of the 9 month IRV swap at $T = 9m$ is $V_n(0, 9m) \times \text{PVBP}_{9m}(T_1, \cdots, T_n)$, and $\text{PVBP}_{9m}(T_1, \cdots, T_n)$ relates to swaps payments starting at time T_1.

The price of the 9 month contract relies on the price of out-of-the-money swaptions that are 9 months into $T_n - 9m$ years, as of time $t = 0$. The price of the 3 month contract is instead calculated through the price of out-of-the-money swaptions with expiry $T = 3m$ and underlying swap starting at $T_0 = 9m$. Typically, swaption markets have $T = T_0$, although theoretically, the pricing framework in this section is still valid even when $T < T_0$ as it is for the 3 month contract in Figure 6. Note that markets may not exist or, at best, be illiquid, for the swaptions needed to price volatility products traded for an expiry trade occurring prior to the beginning of the tenor period of the underlying forward swap rate, $T < T_0$. In this case, the pricing of these products could not be model-free. The simplest solution to price these illiquid assets might rely on the Black's (1976) formula, where one uses as an input the implied volatilities for maturity $T_0 - t$ and replaces, everywhere in the formula, time to maturity $T_0 - t$ with time to maturity $T - t$. Note that we could simply not use swaptions with time to maturity $T - t$ to price these products, if the swaps and, hence, the forward swap rates underlying these swaptions start exist at time T, not at time T_0 as required in Figure 6.

The strategy of Figure 6 eliminates the risk related to the fluctuations of realized interest rate swap volatility, as we now explain. Come time τ, its value is:

$$
\begin{aligned}
&\text{M-Var}_n(0, \tau, 3m) - \text{M-Var}_n(0, \tau, 9m) \\
&= V_n(0, \tau) \times \text{PVBP}_\tau(T_1, \cdots, T_n) + P_\tau(3m)\left[\mathbb{P}_{\text{var},n}(\tau, 3m) - \mathbb{P}_{\text{var},n}(0, 3m)\right] \\
&- \left(V_n(0, \tau) \times \text{PVBP}_\tau(T_1, \cdots, T_n) + P_\tau(9m)\left[\mathbb{P}_{\text{var},n}(0, 9m) - \mathbb{P}_{\text{var},n}(\tau, 9m)\right]\right) \\
&> 0,
\end{aligned}
$$

where we have made use of Eq. (19), and the last inequality follows by the view summarized by Eqs.(23). Note that the strategy eliminates the cumulative realized variance exposure $V_n(0, \tau) \times \text{PVBP}_\tau(T_1, \cdots, T_n)$, as shown by the previous calculations, precisely because we are trading the two IRV swaps in Figure 6, which refer to the volatility of the same forward swap rate at $t = 0$: the forward swap rate of 9 months into $T_n - 9m$ years. As a result, a fixed tenor trade does not need additional positions in the IRV forward contract, as in the case of the rolling tenor trades in Figure 5.

6.2.1.2. Trading through Standardized IRV swap rates

While the trading strategies of the previous section rely on the IRV swap rate of Definition II, trading the Standardized IRV swap rate of Definition III leads to comparable outcomes. For example, suppose that, in analogy with Eq. (21), the view is held that, at some future point $\tau \leq 3m$,

$$\mathbb{P}^*_{\mathrm{var},\hat{n}}(\tau, 3m) > \mathbb{P}^*_{\mathrm{var},\hat{n}}(0, 3m) \quad \text{and} \quad \mathbb{P}^*_{\mathrm{var},n}(\tau, 9m) < \mathbb{P}^*_{\mathrm{var},n}(0, 9m),$$

where the dynamics of the contracts are, now, as in Figure 5. Consider the following strategy: (i) go long the $3m$ contract, (ii) short α units of the $9m$ contract, (iii) short the IRV forward contract of Definition I for the $3m$ maturity, with contract length equal to τ, (iv) go long α units of the IRV forward contract for the $9m$ maturity, with contract length equal to τ. Using the marking to market updates in Eq. (20), we find that the value of this strategy at time τ is:

$$\begin{aligned}
&\text{M-Var}^*_{\hat{n}}(0, \tau, 3m) - \alpha \cdot \text{M-Var}^*_n(0, \tau, 9m) \\
&- V_{\hat{n}}(0, \tau) \times \mathrm{PVBP}_\tau(\hat{T}_1, \cdots, \hat{T}_n) + \alpha \cdot V_n(0, \tau) \times \mathrm{PVBP}_\tau(T_1, \cdots, T_n) \\
&= \mathrm{PVBP}_\tau(\hat{T}_1, \cdots, \hat{T}_n)\left[\mathbb{P}^*_{\mathrm{var},\hat{n}}(\tau, 3m) - \mathbb{P}^*_{\mathrm{var},\hat{n}}(0, 3m)\right] \\
&+ \mathrm{PVBP}_\tau(T_1, \cdots, T_n)\left[\mathbb{P}^*_{\mathrm{var},n}(0, 9m) - \mathbb{P}^*_{\mathrm{var},n}(\tau, 9m)\right] \\
&> 0,
\end{aligned}$$

where α is as in Eq. (22), which ensures the portfolio is worthless at $t = 0$.

Likewise, suppose that, similarly as in Eq. (23), the view is that, at some future point $\tau \leq 3m$,

$$\mathbb{P}^*_{\mathrm{var},n}(\tau, 3m) > \mathbb{P}^*_{\mathrm{var},n}(0, 3m) \quad \text{and} \quad \mathbb{P}^*_{\mathrm{var},n}(\tau, 9m) < \mathbb{P}^*_{\mathrm{var},n}(0, 9m),$$

where the dynamics of the contracts are exactly as in Figure 6. We go long the $3m$ and short the $9m$ Standardized contracts. By Eq. (20), the value of this strategy at time τ is:

$$\begin{aligned}
&\text{M-Var}^*_n(0, \tau, 3m) - \text{M-Var}^*_n(0, \tau, 9m) \\
&= \mathrm{PVBP}_\tau(T_1, \cdots, T_n)\left[\left(\mathbb{P}^*_{\mathrm{var},n}(\tau, 3m) - \mathbb{P}^*_{\mathrm{var},n}(0, 3m)\right) + \left(\mathbb{P}^*_{\mathrm{var},n}(0, 9m) - \mathbb{P}^*_{\mathrm{var},n}(\tau, 9m)\right)\right] \\
&> 0.
\end{aligned}$$

6.2.2 Forward trading

Finally, we consider examples of trading strategies to implement views about the implicit pricing that we can read through the current IRV swap rates. Suppose, for example, that at time $t = 0$, we hold the view that in one year time, the IRV swap rate for a contract expiring in a further year will be greater than the "implied forward" IRV swap rate, i.e. the rate implied by the current term structure of the IRV contracts, viz

$$\mathbb{P}_{\mathrm{var},n}(1, 2) > \mathbb{P}_{\mathrm{var},n}(0, 2) - \mathbb{P}_{\mathrm{var},n}(0, 1). \tag{24}$$

The three IRV swap rates refer to contracts with tenors starting in two year time from $t = 0$, and exhausting at the same time T_n, similarly as for the contracts underlying the fixed tenor trades in Figure 6.

We synthesize the "cheap," implied IRV swap rate, by going long the following portfolio at $t \equiv 0$:

(i) long a two year IRV swap, struck at $\mathbb{P}_{\mathrm{var},n}(0, 2)$ [P.1]
(ii) short a one year IRV swap, struck at $\mathbb{P}_{\mathrm{var},n}(0, 1)$

This portfolio is costless at time zero, and if Eq. (24) holds true in one year, then in one year's time we close (i) and access the payoff in (ii), securing a total payoff equal to:

$$\pi \equiv \text{M-Var}_n\,(0,1,2) + \left[\mathbb{P}_{\text{var},n}\,(0,1) - V_n\,(0,1) \times \text{PVBP}_1\,(T_1,\cdots,T_n)\right].$$

The first term is the market value of the long position in (i) in one year's time, with M-Var$_n\,(0,1,2)$ as in Eq. (19). The second term is the payoff arising from (ii). Using Eq. (19), we obtain,

$$\begin{aligned}\pi &= \left[\mathbb{P}_{\text{var},n}\,(1,2) - \mathbb{P}_{\text{var},n}\,(0,2)\right] P_1\,(2) + \mathbb{P}_{\text{var},n}\,(0,1)\\ &\geq \left[\mathbb{P}_{\text{var},n}\,(1,2) - \mathbb{P}_{\text{var},n}\,(0,2) + \mathbb{P}_{\text{var},n}\,(0,1)\right] P_1\,(2)\\ &> 0,\end{aligned}$$

where the first inequality follows by $P_1\,(2) < 1$, and the second holds by Eq. (24).

Likewise, we can implement trading strategies consistent with views about implied Standardized IRV swap rates. Suppose we anticipate that:

$$\mathbb{P}^*_{\text{var},n}\,(1,2) > \mathbb{P}^*_{\text{var},n}\,(0,2) - \mathbb{P}^*_{\text{var},n}\,(0,1),$$

where the dynamics of the contracts are again as in Figure 6. Consider the same portfolio as in [P.1] with Standardized IRV contracts replacing IRV contracts. In one year's time, the value of this portfolio is:

$$\begin{aligned}&\text{M-Var}^*_n\,(0,1,2) + \left[\mathbb{P}^*_{\text{var},n}\,(0,1) - V_n\,(0,1)\right] \times \text{PVBP}_1\,(T_1,\cdots,T_n)\\ &= \left[\mathbb{P}^*_{\text{var},n}\,(1,2) + \mathbb{P}^*_{\text{var},n}\,(0,1) - \mathbb{P}^*_{\text{var},n}\,(0,2)\right] \times \text{PVBP}_1\,(T_1,\cdots,T_n)\\ &> 0,\end{aligned}$$

where we have used the marking to market update in Eq. (20).

7. Swap volatility indexes

7.1. *Percentage*

The price of the IRV forward agreement of Definition I can be normalized by the length of the contract, $T-t$, and quoted in terms of the current swap's PVBP, so as to lead to the following index of expected volatility:

$$\boxed{\text{IRS-VI}_n\,(t,T) = \sqrt{\frac{1}{T-t}\mathbb{P}^*_{\text{var},n}\,(t,T)}} \tag{25}$$

where $\mathbb{P}^*_{\text{var},n}\,(t,T)$ is the Standardized IRV swap rate in Eq. (10). We label the index in Eq. (25) "IRS-VI."

In Appendix C.1, we show that the expression in Eq. (25) can be simplified and fed with only the Black's (1976) volatilities on the out-of-the money swaption skew (see Eq. (C.5)). In the Appendix, we also show that if $V_n\,(t,T)$ and the path of the short-term rate in the time interval $[t,T]$ were uncorrelated, the IRS-VI in Eq. (25) would be the risk-neutral expectation of the future realized variance defined as:

$$\text{E-Vol}_n\,(t,T) \equiv \sqrt{\frac{1}{T-t}E_Q\left[V_n\,(t,T)\right]}, \tag{26}$$

where E_Q denotes expectation in a risk-neutral market.

In practice, $V_n(t,T)$ and interest rates are likely to be correlated. For example, if the short-term rate is generated by the Vasicek (1977) model, the correlation between $V_n(t,T)$ and $\mathrm{PVBP}_T(T_1,\cdots,T_n)$ is positive—a finding we established as a by-product of experiments reported in Appendix F. Despite this theoretical dependence, Eq. (25) is, for all intents and purposes, the same as the risk-neutral expectation of future realized volatility, Eq. (26). Figure 7 plots the relation between volatility and the forward rate arising within the Vasicek market. The left panels plot the relation between the forward swap rate and: (i) the $\text{IRS-VI}_n(t,T)$ index in Eq. (25), for a maturity $T-t=1$ month and tenor $n=5$ years (top panels), a maturity $T-t=1$ year and tenor $n=10$ years (bottom panels), and regular quarterly reset dates; and (ii) the risk-neutral expectation of future realized volatility of the forward swap rate, computed through Eq. (26). The right panels plot the relation between the forward swap rate and the instantaneous volatility of the forward swap rate, as defined in Appendix F, Eq. (F.4). Appendix F provides details regarding these calculations, as well as additional numerical results relating to a number of alternative combinations of maturities and tenors (see Table A.1). In these numerical examples, the IRS-VI index provides an upper bound to the expected volatility in a risk-neutral market, and it highly correlates with it. The IRS-VI and expected volatility respond precisely in the same way to changes in market conditions, as summarized by movements in the forward swap rate, and deviate quite insignificantly from each other. Interestingly, the Vasicek market is one where we observe the interest rate counterpart of the "leverage effect" observed in equity markets: low interest rates are associated with high interest rate volatility.

7.2. *Basis point*

The BP counterpart to the IRS-VI index in Eq. (25) is:

$$\boxed{\text{IRS-VI}_n^{\mathrm{BP}}(t,T)=\sqrt{\frac{1}{T-t}\mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T)},\quad \mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T)=\frac{\mathbb{F}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}} \tag{27}$$

where $\mathbb{F}^{\mathrm{BP}}_{\mathrm{var},n}(t,T)$ is the price of the IRV forward agreement in Eq. (14). Similar to the percentage index in Eq. (25), the Basis Point index in Eq. (27) can be calculated by feeding Black's (1976) formula with the swaption skew (see Eq. (G.5) in Appendix G.1).

Interestingly, our model-free BP volatility index squared and rescaled by $\frac{1}{T-t}$ coincides with the conditional second moment of the forward swap rate, as it turns out by comparing our formula in Eq. (27) with expressions in Trolle and Schwartz (2011). It is an interesting statistical property, which complements the asset pricing foundations lied down in Sections 4 and 5, pertaining to security designs, hedging, replication, and those relating to marking to market and trading strategies in Section 6. In Section 9, we shall show that our Basis Point index enjoys the additional interesting property of being resilient to jumps: its value remains the same, and hence model-free, even in the presence of jumps.

Finally, Figure 8 reports experiments that are the basis point counterparts to those in Figure 7. In these experiments, we compare BP volatility as referenced by our index for five and ten year tenors, with that arising in a risk-neutral Vasicek market, as well as the instantaneous BP volatility of the forward rate, as predicted by the Vasicek model. Once again, the index and expected volatility are practically the same, a property arising in a number of additional experiments reported in Appendix G.4.

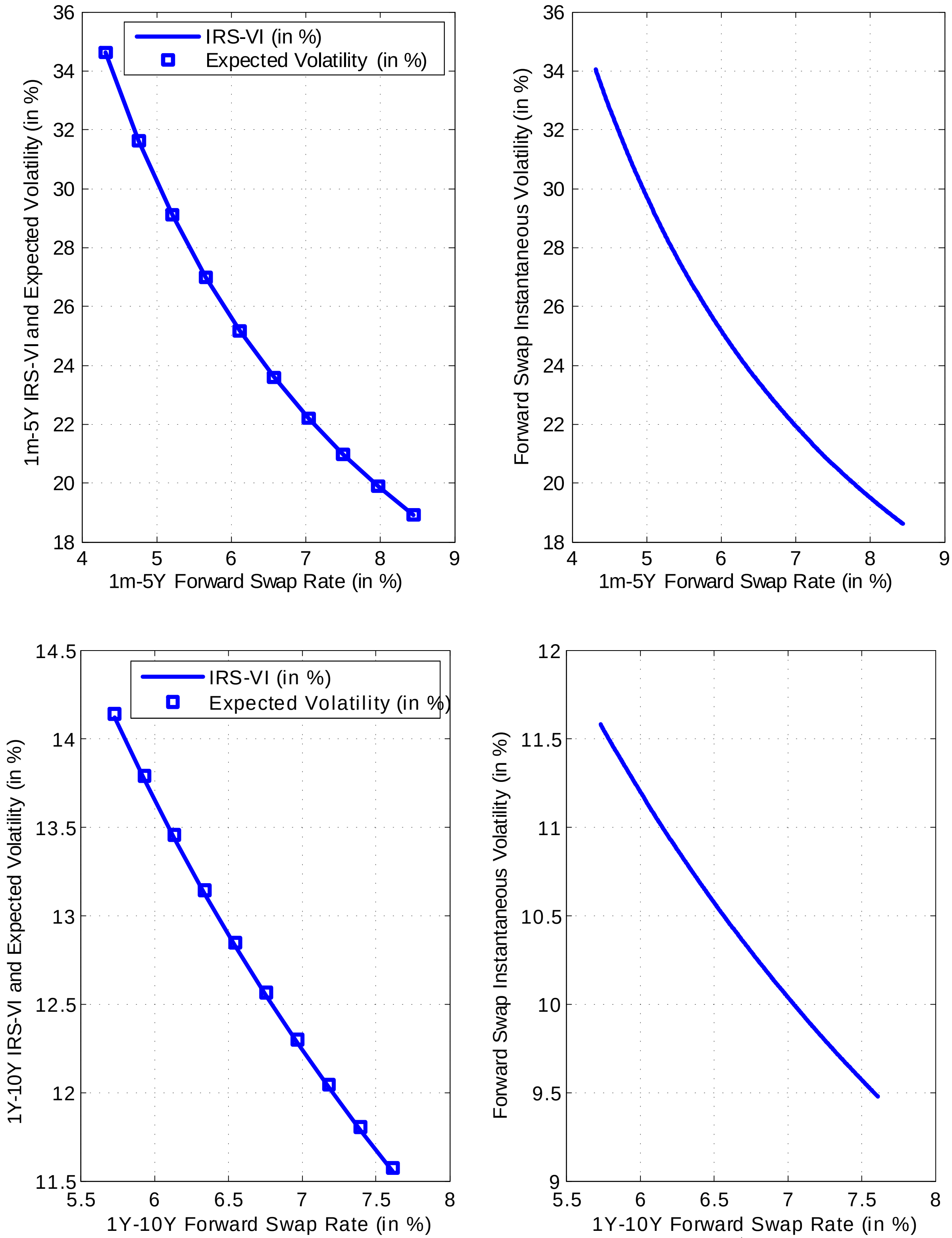

FIGURE 7. Left panels: The IRS-VI index and expected volatility (both in percentage) in a risk-neutral Vasicek market, as a function of the level of the forward swap rate. Right panels: the instantaneous forward swap rate volatility (in percentage) as a function of the forward swap rate. Top panels relate to a length of the variance contract equal to 1 month and tenor of 5 years. Bottom panels relate to a length of the variance contract equal to 1 year and tenor of 10 years.

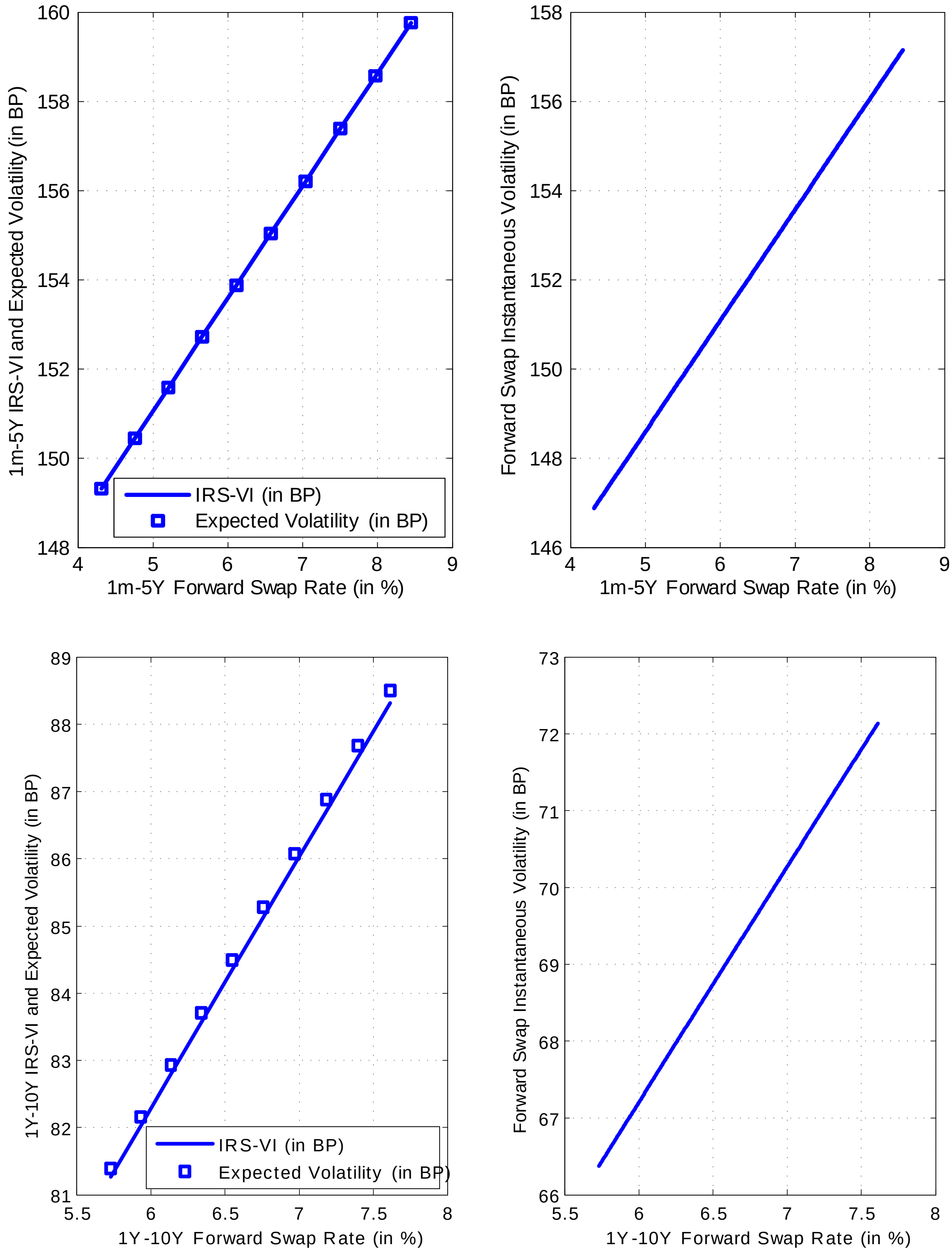

FIGURE 8. Left panels: The IRS-VI index and expected volatility (both in Basis Points) in a risk-neutral Vasicek market, as a function of the level of the forward swap rate. Right panels: the instantaneous forward swap rate volatility (in Basis Points) as a function of the forward swap rate. Top panels relate to a length of the variance contract equal to 1 month and tenor of 5 years. Bottom panels relate to a length of the variance contract equal to 1 year and tenor of 10 years.

8. Rates versus equity variance contracts and indexes

The pricing framework underlying IRV contracts and indexes relies on spanning arguments, such as those in Bakshi and Madan (2000) and Carr and Madan (2001), as is also the case with equity variance contracts and indexes. However, IRV contracts and indexes quite differ from the equity case for a number of reasons.

First, the payoffs of IRV contracts introduced in this paper have two sources of uncertainty: one related to the realized variance of the forward swap rate, and another related to the forward PVBP. These features of the contract design and the assumption that swap rates are stochastic distinguish our methodology from the work that has been done on pricing equity volatility. In the equity case, the volatility index is:

$$\mathrm{VIX}(t,T) = \sqrt{\frac{1}{T-t}\frac{2}{e^{-\bar{r}(T-t)}}\left[\sum_{i:K_i<F_t(T)}\frac{\mathrm{Put}_t(K_i,T)}{K_i^2}\Delta K_i + \sum_{i:K_i\geq F_t(T)}\frac{\mathrm{Call}_t(K_i,T)}{K_i^2}\Delta K_i\right]}, \tag{28}$$

where $\bar{r}$ is the instantaneous interest rate, *assumed to be constant*, and $\mathrm{Put}_t(K,T)$ and $\mathrm{Call}_t(K,T)$ are the market prices as of time t of out-of-the money European put and call equity options with strike prices equal to K and time to maturity $T-t$, $F_t(T)$ is the forward price of equity, and ΔK_i are as in Eq. (8). Note that due to constant discounting, the fair strike for an equity variance contract is the same as the index, up to time-rescaling and squaring, and is given by:

$$\mathbb{P}_{\text{equity}}(t,T) \equiv (T-t)\cdot \mathrm{VIX}^2(t,T). \tag{29}$$

As for IRV contracts, note that the fair value of the Standardized IRV swap rate, $\mathbb{P}^*_{\text{var},n}(t,T)$ in Eq. (7), is still the index, up to time-rescaling and squaring and, by Eq. (25), is given by:

$$\text{IRS-VI}_n(t,T) = \sqrt{\frac{1}{T-t}\frac{2}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}\left[\sum_{i:K_i<R_t}\frac{\mathrm{SWPN}_t^{\mathrm{R}}(K_i,T;T_n)}{K_i^2}\Delta K_i + \sum_{i:K_i\geq R_t}\frac{\mathrm{SWPN}_t^{\mathrm{P}}(K_i,T;T_n)}{K_i^2}\Delta K_i\right]}. \tag{30}$$

While the two indexes in Eqs. (28) and (30) aggregate prices of out-of-the money derivatives using the same weights, they differ for three reasons. (i) the VIX in Eq. (28) is constructed by rescaling a weighted average of out-of-the money option prices through the inverse of the price of a zero with the same expiry date as that of the variance contract in a market with constant interest rates, $e^{-\bar{r}(T-t)}$. Instead, the IRS-VI index in Eq. (30) rescales a weighted average of out-of-the money swaption prices with the inverse of the price of a basket of bonds, $\mathrm{PVBP}_t(T_1,\cdots,T_n)$, each of them expiring at the end-points of the swap's fixed payment dates; (ii) the different nature of the derivatives involved in the definition of the two indexes: European options for the $\mathrm{VIX}(t,T)$ index, and swaptions for the $\text{IRS-VI}_n(t,T)$ index; (iii) the extra dimension in swap rate volatility introduced by the length of the tenor period underlying the forward swap rate, $T_n - T_0$.

There are further distinctions to be made. Consider the current value of a forward for delivery of the rate variance at time T in Section 4.1 (Definition I), which by Eq. (8) is:

$$\mathbb{F}_{\text{var},n}(t,T) = \mathrm{PVBP}_t(T_1,\cdots,T_n)\cdot \mathbb{P}_{\text{var},n}(t,T). \tag{31}$$

As for equity, the price to be paid at t, for delivery of equity volatility at T, is, instead:

$$\mathbb{F}_{\text{equity}}(t,T) \equiv P_t(T) \cdot \mathbb{P}_{\text{equity}}(t,T). \tag{32}$$

The two prices, $\mathbb{F}_{\text{var},n}$ and $\mathbb{F}_{\text{equity}}$, scale up to the time t fair values of their respective notionals at time T: (i) the fair value $\text{PVBP}_t(T_1,\cdots,T_n)$ of the *random notional* $\text{PVBP}_T(T_1,\cdots,T_n)$, in Eq. (31); and (ii) the fair value $P_t(T)$ of the *deterministic notional* of one dollar, in Eq. (32). This remarkable property arises in spite of the different assumptions underlying the two contracts: (i) *random* interest rates, for the rate variance contract, and (ii) *constant* interest rates, for the equity variance contract.

Finally, consider the Basis Point IRV contract in Section 5. By using the expression in Eq. (14) for the fair value of the Basis Point Standardized IRV swap rate, the BP index in Eq. (27) can be written as:

$$\begin{aligned}&\text{IRS-VI}_n^{\text{BP}}(t,T)\\&=\sqrt{\frac{1}{T-t}\frac{2}{\text{PVBP}_t(T_1,\cdots,T_n)}\left[\sum_{i:K_i<R_t}\text{SWPN}_t^{\text{R}}(K_i,T;T_n)\,\Delta K_i+\sum_{i:K_i\geq R_t}\text{SWPN}_t^{\text{P}}(K_i,T;T_n)\,\Delta K_i\right]}.\end{aligned} \tag{33}$$

For the index of the Basis Point volatility metric in Eq. (33), the weights to be given to the out-of-the money swaption prices are not inversely proportional to the square of the strike, as in the case for the equity volatility index in Eq. (28). In the theoretical case where we were given a continuum of strikes, our BP index would actually be an *equal-weighted* average of out-of-the money swaption prices, as revealed by the exact expression of the BP-Standardized IRV swap rate in Appendix G (see Eq. (G.4)), and consistently with the constant Gaussian vega explanations of Section 5.1 (see Eq. (17))—although due to discretization and truncation, the weights in Eq. (33), ΔK_i, might not be exactly the same. Therefore, the Basis Point interest rate volatility index and contracts differ from equity, not only due to the aspects pointed out for the "percentage" volatility, but also for the particular weighting each swaption price enters into the definition of the index and contracts. Note that these different weightings reflect different hedging strategies. As explained in Sections 4 and 5, replicating a variance contract agreed over a "percentage" volatility metric requires relying on "log-contracts," whereas replicating a variance contract agreed over the "basis point" realized variance, requires relying on "quadratic contracts."

9. Resilience to jumps

This section examines the pricing and indexing of expected interest rate swap volatility in markets where the forward swap rate is a jump-diffusion process with stochastic volatility:

$$\begin{aligned}\frac{dR_\tau(T_1,\cdots,T_n)}{R_\tau(T_1,\cdots,T_n)} &= -\left(\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta(\tau)\right)d\tau\\&\quad+\sigma_\tau(T_1,\cdots,T_n)\cdot dW^*(\tau)+\left(e^{j_n(\tau)}-1\right)dN^*(\tau), \qquad \tau\in[t,T],\end{aligned} \tag{34}$$

where $W^*(\tau)$ is a multidimensional Wiener process defined under the *swap probability* Q_{swap}, $\sigma_\tau(T_1,\cdots,T_n)$ is a diffusion component, adapted to $W^*(\tau)$, $N^*(\tau)$ is a Cox process under

Q_{swap} with intensity equal to $\eta(\tau)$, and $j_n(\tau)$ is the logarithmic jump size. (See, e.g., Jacod and Shiryaev (1987, p. 142-146), for a succinct discussion of jump-diffusion processes.) By applying Itô's lemma for jump-diffusion processes to Eq. (34), we have that

$$d\ln R_\tau(T_1,\cdots,T_n) = (\cdots)\,d\tau + \sigma_\tau(T_1,\cdots,T_n)\cdot dW^*(\tau) + j_n(\tau)\,dN^*(\tau),$$

such that the realized variance of the *logarithmic changes* of the forward swap rate over a time interval $[t,T]$, or *percentage variance*, is now:

$$V_n^J(t,T) \equiv \int_t^T \|\sigma_\tau(T_1,\cdots,T_n)\|^2\,d\tau + \int_t^T j_n^2(\tau)\,dN^*(\tau). \tag{35}$$

Instead, the realized variance of the *arithmetic changes* of the forward swap rate over $[t,T]$, or *basis point variance*, is:

$$V_n^{J,\text{BP}}(t,T) \equiv \int_t^T R_\tau^2(T_1,\cdots,T_n)\,\|\sigma_\tau(T_1,\cdots,T_n)\|^2\,d\tau + \int_t^T R_\tau^2(T_1,\cdots,T_n)\left(e^{j_n(\tau)}-1\right)^2 dN^*(\tau). \tag{36}$$

These definitions generalize those in Eq. (5) and Eq. (12).

We now derive the fair value of the IRV Standardized contracts in Sections 4 and 5 in this new setting. In Appendix H, we show that the fair value of the *percentage* IRV contract payoff Var-Swap$_n^*(t,T)$ in Eq. (7), $\mathbb{P}_{\text{var},n}^{J,*}(t,T)$ say, generalizes that in Eq. (10) because of the presence of one additional term, capturing jumps, as follows:

$$\mathbb{P}_{\text{var},n}^{J,*}(t,T) = \mathbb{P}_{\text{var},n}^{*}(t,T) - 2\mathbb{E}_{Q_{\text{swap},\tau}}\left[\int_t^T \left(e^{j_n(\tau)} - 1 - j_n(\tau) - \frac{1}{2}j_n^2(\tau)\right) dN^*(\tau)\right], \tag{37}$$

where $\mathbb{P}_{\text{var},n}^{*}(t,T)$ is the fair value of the Standardized IRV contract in Eq. (10). Suppose, for example, that the distribution of jumps is skewed towards negative values. The fair value $\mathbb{P}_{\text{var},n}^{J,*}(t,T)$ should then be higher than it would be in the absence of jumps. Assume, in particular, that the distribution of jumps collapses to a single point, $\bar{j}<0$ say, and that the jump intensity equals some positive constant $\bar{\eta}$, in which case the fair value in Eq. (37) collapses to,

$$\mathbb{P}_{\text{var},n}^{J,*}(t,T) = \mathbb{P}_{\text{var},n}^{*}(t,T) + 2(T-t)\cdot\bar{\eta}\mathcal{J}, \tag{38}$$

where we have defined the positive constant $\mathcal{J} \equiv -(e^{\bar{j}} - 1 - \bar{j} - \frac{1}{2}\bar{j}^2) > 0$. The index of percentage swap volatility is now,

$$\boxed{\text{IRS-VI}_n^J(t,T) \equiv \sqrt{\text{IRS-VI}_n^2(t,T) - \frac{2}{T-t}\mathbb{E}_{Q_{\text{swap},\tau}}\left(\int_t^T \left(e^{j_n(\tau)} - 1 - j_n(\tau) - \tfrac{1}{2}j_n^2(\tau)\right) dN^*(\tau)\right)}} \tag{39}$$

where $\text{IRS-VI}_n(t,T)$ is as in Eq. (25). In the special case of the parametric example underlying the price of the IRV contract of Eq. (38), the previous formula reduces to,

$$\text{IRS-VI}_n^J(t,T) = \sqrt{\text{IRS-VI}_n^2(t,T) + 2\bar{\eta}\mathcal{J}}.$$

In a notable contrast, for the case of basis point IRV, we prove in Appendix H that the fair value of the basis point IRV contract payoff in Definition IV-(c), denoted $\mathbb{P}_{\text{var},n}^{J,*\text{BP}}(t,T)$, is resilient to jumps, meaning

$$\mathbb{P}_{\text{var},n}^{J,*\text{BP}}(t,T) = \mathbb{P}_{\text{var},n}^{*\text{BP}}(t,T), \tag{40}$$

where $\mathbb{P}^{*\mathrm{BP}}_{\mathrm{var},n}(t,T)$ is as in Eq. (15). Accordingly, the basis point index of swap volatility, IRS-$\mathrm{VI}_n^{\mathrm{BP},J}(t,T)$ say, is the same as that we derived in the absence of jumps,

$$\boxed{\text{IRS-VI}_n^{\mathrm{BP},J}(t,T) = \text{IRS-VI}_n^{\mathrm{BP}}(t,T)} \tag{41}$$

where IRS-$\mathrm{VI}_n^{\mathrm{BP}}(t,T)$ is as in Eq. (27).

10. Index implementation

This section provides examples of a numerical implementation of the index. Section 10.1 develops a step-by-step illustration of the index calculations, and Section 10.2 contains examples of the index behavior around particular dates.

10.1. A numerical example

The following example is a non-limiting illustration of the main steps involved into the construction of the IRS-VI indexes IRS-$\mathrm{VI}_n^{\mathrm{BP}}(t,T)$ and IRS-$\mathrm{VI}_n(t,T)$ in Eq. (25) and Eq. (27). We utilize hypothetical data for implied volatilities expressed in percentage terms for swaptions maturing in one month and tenor equal to five years. The first two columns of Table V reports strike rates, K say, and the "skew," i.e. the percentage implied volatilities for each strike rate, denoted as $\mathrm{IV}(K)$ ("Percentage Implied Vol"). For reference, the third column provides basis point implied volatilities for each strike rate K, denoted as $\mathrm{IV}^{\mathrm{BP}}(K)$, and computed as:

$$\mathrm{IV}^{\mathrm{BP}}(K) = \mathrm{IV}(K) \cdot R, \tag{42}$$

where R denotes the current forward swap rate ("Basis Point Implied Vol"). For example, from Table V, $R = 2.7352\%$ and $\mathrm{IV}(K)\big|_{K=R} = 35.80\%$ and, then, $\mathrm{IV}^{\mathrm{BP}}(K)\big|_{K=R} = 2.7352\% \times 35.80\% = 0.979202\%$, which is 97.9202 basis points volatility. Basis point volatilities are not needed to compute the indexes of this paper.

The two IRS-VI indexes in Eq. (25) and Eq. (27) are implemented through Black's (1976) formula, as explained in Appendix C.1, Eq. (C.5), and Appendix G.1, Eq. (G.5). First, we plug the skew $\mathrm{IV}(K)$ into the Black's (1976) formulae,

$$\begin{aligned}
\hat{Z}\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right) &= Z\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right) + K - R, && (43)\\
Z(R,T,K;V) &= R\Phi(d) - K\Phi(d-\sqrt{V}), \quad d = \frac{\ln\frac{R}{K} + \frac{1}{2}V}{\sqrt{V}}, && (44)
\end{aligned}$$

where Φ denotes the cumulative standard normal distribution, and $\hat{Z}(\cdot)$ and $Z(\cdot)$ are the Black's prices, i.e. swaption prices (receivers, for $\hat{Z}(\cdot)$; and payers, for $Z(\cdot)$), divided by the PVBP. Second, we use Eqs. (43)-(44), and calculate two indexes in Eq. (25) and Eq. (27):

$$\begin{aligned}
&\text{IRS-VI}_n(t,T)\\
&= 100 \times \sqrt{\frac{2}{T-t}\left[\sum_{i:K_i<R_t} \frac{\hat{Z}\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right)}{K_i^2}\Delta K_i + \sum_{i:K_i\geq R_t} \frac{Z\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right)}{K_i^2}\Delta K_i\right]}, \qquad (45)
\end{aligned}$$

and:

$$\text{IRS-VI}_n^{\mathrm{BP}}(t,T)$$

$$= 100 \times 100 \times \sqrt{\frac{2}{T-t}\left[\sum_{i:K_i<R_t} \hat{Z}\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right)\Delta K_i + \sum_{i:K_i\geq R_t} Z\left(R,T,K_i;(T-t)\,\mathrm{IV}^2(K_i)\right)\Delta K_i\right]}. \tag{46}$$

The basis point index in Eq. (46), is rescaled by 100^2, to mimic the market practice to express basis point implied volatility as the product of rates times log-volatility, where both rates and log-volatility are multiplied by 100.

The fourth column of Table V provides the values of $\hat{Z}$ and Z ("Black's prices"), for each strike rate.

Table V

Strike Rate (%)	Percentage Implied Vol	Basis Point Implied Vol	Black's prices: Receiver Swaption ($\hat{Z}$)	Black's prices: Payer Swaption Z
1.7352	36.1900	98.9869	≈ 0	$10.0000\cdot10^{-3}$
1.9852	36.1900	98.9869	$0.0007\cdot10^{-3}$	$7.5007\cdot10^{-3}$
2.2352	36.1200	98.7954	$0.0259\cdot10^{-3}$	$5.0259\cdot10^{-3}$
2.4352	35.9900	98.4398	$0.1773\cdot10^{-3}$	$3.1773\cdot10^{-3}$
2.5352	35.9300	98.2757	$0.3692\cdot10^{-3}$	$2.3692\cdot10^{-3}$
2.6352	35.8600	98.0843	$0.6793\cdot10^{-3}$	$1.6793\cdot10^{-3}$
2.6852	35.8300	98.0022	$0.8855\cdot10^{-3}$	$1.3855\cdot10^{-3}$
2.7352	35.8000	97.9202	$1.1272\cdot10^{-3}$	$1.1272\cdot10^{-3}$
2.7852	35.7600	97.8108	$1.4037\cdot10^{-3}$	$0.9037\cdot10^{-3}$
2.8352	35.7300	97.7287	$1.7142\cdot10^{-3}$	$0.7142\cdot10^{-3}$
2.9352	35.6700	97.5646	$2.4270\cdot10^{-3}$	$0.4270\cdot10^{-3}$
3.0352	35.6000	97.3731	$3.2406\cdot10^{-3}$	$0.2406\cdot10^{-3}$
3.2352	35.4700	97.0175	$5.0644\cdot10^{-3}$	$0.0644\cdot10^{-3}$
3.4852	35.3100	96.5799	$7.5092\cdot10^{-3}$	$0.0092\cdot10^{-3}$
3.7352	35.1400	96.1149	$10.0010\cdot10^{-3}$	$0.0010\cdot10^{-3}$

Table VI provides details regarding the computation of the indexes $\text{IRS-VI}_n^{\text{BP}}(t,T)$ and $\text{IRS-VI}_n(t,T)$ through Eqs. (46) and (45): the second column displays the type of out-of-the money swaption entering into the calculation; the third column has the corresponding Black's prices; the fourth and fifth columns report the weights each price bears towards the final computation of the index before the final rescaling of $\frac{2}{T-t}$; finally, the sixth and seventh columns report each out-of-the money swaption price multiplied by the appropriate weight (i.e. third column multiplied by the fourth column for the "Basis Point Contribution " , and third column multiplied by the fifth column for the "Percentage Contribution."

Table VI

Strike Rate (%)	Swaption Type	Price	Weights: Basis Point ΔK_i	Weights: Percentage $\Delta K_i/K_i^2$	Contributions to Strikes: Basis Point Contribution	Contributions to Strikes: Percentage Contribution
1.7352	Receiver	≈ 0	0.0025	8.3031	≈ 0	≈ 0
1.9852	Receiver	$0.0007\cdot10^{-3}$	0.0025	6.3435	$0.0018\cdot10^{-6}$	$0.0046\cdot10^{-3}$
2.2352	Receiver	$0.0259\cdot10^{-3}$	0.0022	4.5035	$0.0583\cdot10^{-6}$	$0.1167\cdot10^{-3}$
2.4352	Receiver	$0.1773\cdot10^{-3}$	0.0015	2.5294	$0.2660\cdot10^{-6}$	$0.4485\cdot10^{-3}$
2.5352	Receiver	$0.3692\cdot10^{-3}$	0.0010	1.5559	$0.3692\cdot10^{-6}$	$0.5744\cdot10^{-3}$
2.6352	Receiver	$0.6793\cdot10^{-3}$	0.0008	1.0800	$0.5095\cdot10^{-6}$	$0.7337\cdot10^{-3}$
2.6852	Receiver	$0.8855\cdot10^{-3}$	0.0005	0.6935	$0.4428\cdot10^{-6}$	$0.6141\cdot10^{-3}$
2.7352	ATM	$1.1272\cdot10^{-3}$	0.0005	0.6683	$0.5636\cdot10^{-6}$	$0.7533\cdot10^{-3}$
2.7852	Payer	$0.9037\cdot10^{-3}$	0.0005	0.6446	$0.4518\cdot10^{-6}$	$0.5825\cdot10^{-3}$
2.8352	Payer	$0.7142\cdot10^{-3}$	0.0007	0.9330	$0.5357\cdot10^{-6}$	$0.6664\cdot10^{-3}$
2.9352	Payer	$0.4270\cdot10^{-3}$	0.0010	1.1607	$0.4270\cdot10^{-6}$	$0.4956\cdot10^{-3}$
3.0352	Payer	$0.2406\cdot10^{-3}$	0.0015	1.6282	$0.3609\cdot10^{-6}$	$0.3917\cdot10^{-3}$
3.2352	Payer	$0.0644\cdot10^{-3}$	0.0023	2.1497	$0.1448\cdot10^{-6}$	$0.1384\cdot10^{-3}$
3.4852	Payer	$0.0092\cdot10^{-3}$	0.0025	2.0582	$0.0229\cdot10^{-6}$	$0.0188\cdot10^{-3}$
3.7352	Payer	$0.0010\cdot10^{-3}$	0.0025	1.7919	$0.0024\cdot10^{-6}$	$0.0017\cdot10^{-3}$
				SUMS	$4.1567\cdot10^{-6}$	$5.5405\cdot10^{-3}$

The two indexes are computed as

$$\text{IRS-VI}_n(t,T) = 100 \times \sqrt{\frac{2}{12^{-1}} \times 5.5405 \cdot 10^{-3}} = 36.4653,$$

$$\text{IRS-VI}_n^{\text{BP}}(t,T) \quad = \quad 100 \times 100 \times \sqrt{\frac{2}{12^{-1}} \times 4.1567 \cdot 10^{-6}} = 99.8803.$$

In comparison, ATM implied volatilities are $\text{IV}(R) = 35.8000$ and $\text{IV}^{\text{BP}}(R) = 97.9202$.

10.2. Historical performance

This section documents the performance of the IRS-VI index over selected days by relying on data provided by a major interdelear broker.

- First, we consider the index behavior at four selected dates: (i) February 16, 2007, (ii) February 15, 2008, (iii) February 13, 2009, and (iv) February 12, 2010.
- Second, we document the index behavior over the Lehman's collapse occurred in September 2008.

We calculate the index through Eq. (25), using ATM swaptions and out-of-the-money (OTM, henceforth) swaptions, with moneyness equal to ∓ 200, ∓ 100, ∓ 50, and, finally, ∓ 25 basis points away from the current forward swap rate. We report the index and implied volatilities in percentages. "Basis point volatilities" are percentage implied volatilities multiplied by the current forward swap rate. The Basis Point IRS-VI index of Section 5, $\text{IRS-VI}_n^{\text{BP}}(t,T)$, defined in Eq. (27), aims to aggregate information about these volatilities, in that it tracks expected volatility cast in basis point terms. We only report experiments relating to the percentage index, as defined in Eq. (25), $\text{IRS-VI}_n(t,T)$. Furthermore, note that the index is calculated using rolling tenors, as for the trading strategy in Figure 9: for each time-to-maturity T, $\text{IRS-VI}_n(t,T)$ is the volatility index for tenor period equal to $T_n - T$, for all considered n.

Figures 9 through 12 compare the IRS-VI index with the implied volatility for at-the-money swaptions with five year tenors (ATM volatility, henceforth), as a function of the maturity of the variance contract (one, three, six, nine and twelve months), over the selected dates. Figures 13 through 16 plot the IRS-VI "surface" for the same dates, that is, the plot of the IRS-VI index against (i) the maturity of the variance contract, and (ii) the tenor of the forward rate.

Figures 9-12 show that quantitatively, the IRS-VI index behaves dissimilarly from ATM volatilities. It may be larger or smaller than ATM volatilities, depending on the specific date at which it is measured or the horizon of the variance contract. For example, on February 15, 2008, the IRS-VI index is higher than ATM volatilities for low maturities of the variance contract, and is lower otherwise. The reason for these differences is that the IRS-VI aggregates information relating to both ATM and OTM volatilities, in such a way to isolate expected volatility from other factors, such as changes in the underlying forward rates. ATM volatilities, instead, and by default, only rely on the current forward swap rates. On February 12, 2010, to cite a second example, OTM volatilities were in general higher (resp. lower) than ATM volatilities, in correspondence of smaller (larger) maturities. This pattern is captured by the IRS-VI, which is higher than ATM volatilities for maturities up to six months, and lower than ATM volatilities for maturities of nine and twelve months.

Qualitatively, the shape of the IRS-VI index displays more pronounced characteristics than ATM volatilities, when assessed against the length of the variance contract. In two days occurring during "bad times" (February 15, 2008 and February 13, 2009), the IRS-VI slopes downwards more sharply than ATM volatilities do, thereby vividly signaling the market expectation that bad times will be followed by periods where resolution of uncertainty about interest rates will occur. In good times (February 16, 2007), the IRS-VI index and ATM volatilities both slope up, although moderately, reflecting the market concern these good times, while persistent, will be slightly more uncertain in the future; even in this simple case, the IRS-VI index exhibits a richer behavior than ATM volatilities, as it flattens out for maturities higher than nine months. Finally, the hump shape of the IRS-VI index and ATM volatilities occurring on February 12, 2010 reflects the market expectation that the uncertainty about interest rates will be rising in the immediate, although then it will subside in the following months.

Naturally, adding to the fact the IRS-VI index and ATM volatilities are simply not the same thing, both quantitatively and qualitatively, variance contracts such as those in Section 4 cannot be priced using solely ATM volatilities. Rather, these contracts need to rely upon the IRS-VI index, as explained in the previous sections. Figures 13 through 16 show how the IRS-VI index can be used to express views about developments in interest rate swap volatility. In fact, one of the added features of the IRS-VI index and contract, compared to the VIX® index for equities, is the possibility for market participants to choose the tenor of the forward swap rate, on top of the period length about which to convey volatility views. This added dimension arises quite naturally, as a result of an increased complexity of fixed income derivatives, as opposed to equity. As an example, the trading strategies of Figure 5 can be implemented on a particular tenor period $T_n - T$, and trading can take place over one of the tenors in Figure 13-16, where the mispricing is thought to be occurring.

Figures 17 through 21 depict the behavior of the index around the Lehman's collapse, occurred on September 15, 2008. A few days before the collapse, the index was downward sloping with respect to maturity, reflecting market expectations of possibly imminent negative tail events. These expectations would reinforce the day of the Lehman's collapse, and only partially weaken over the week of this event.

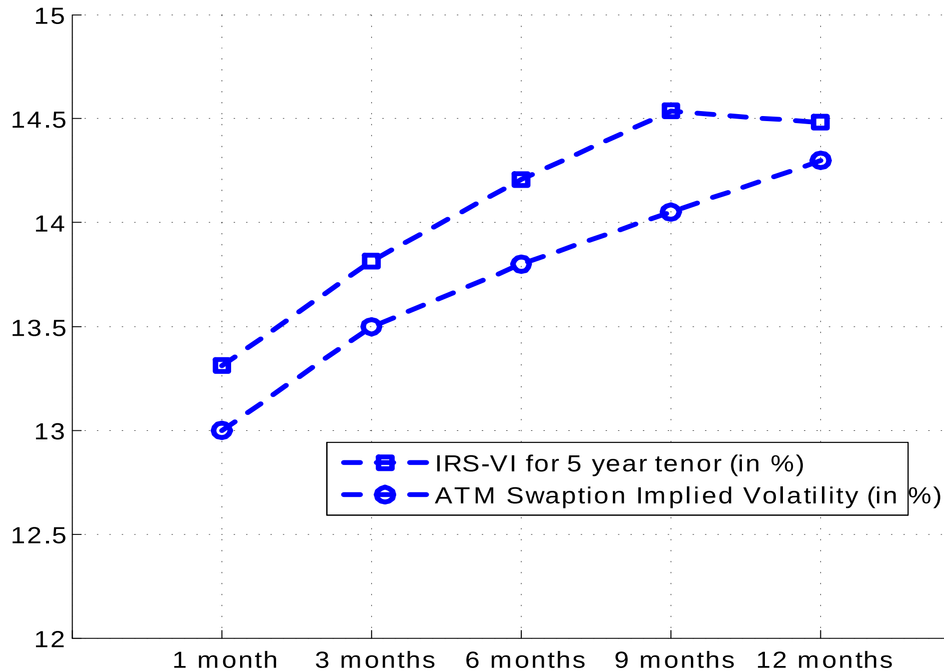

FIGURE 9. Term structure of the Interest Rate Swap Volatility Index (IRS-VI)—February 16, 2007.

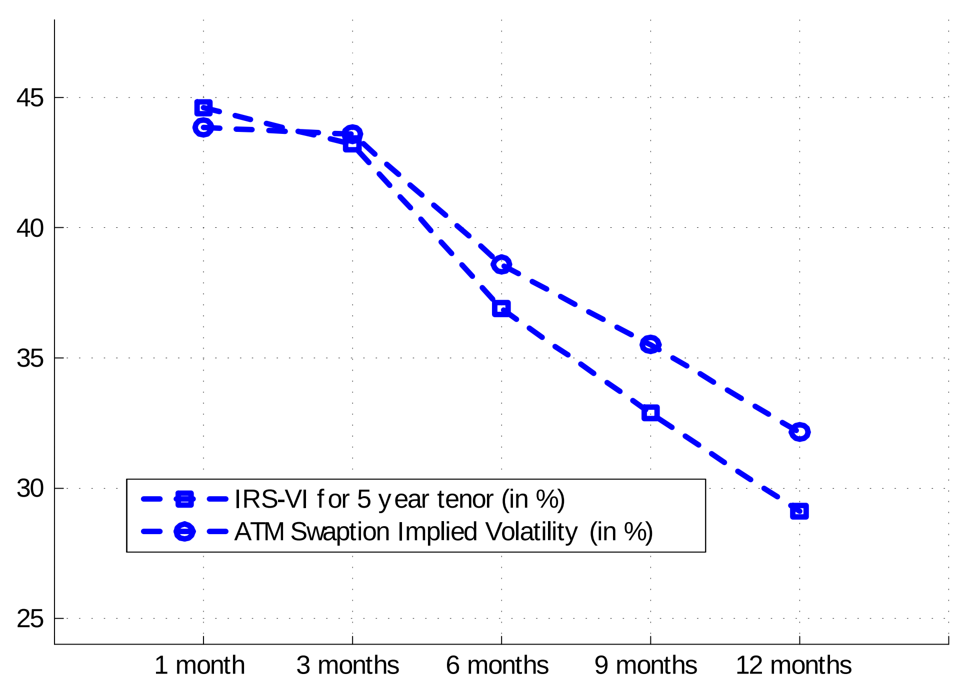

FIGURE 10. Term structure of the Interest Rate Swap Volatility Index (IRS-VI)—February 15, 2008.

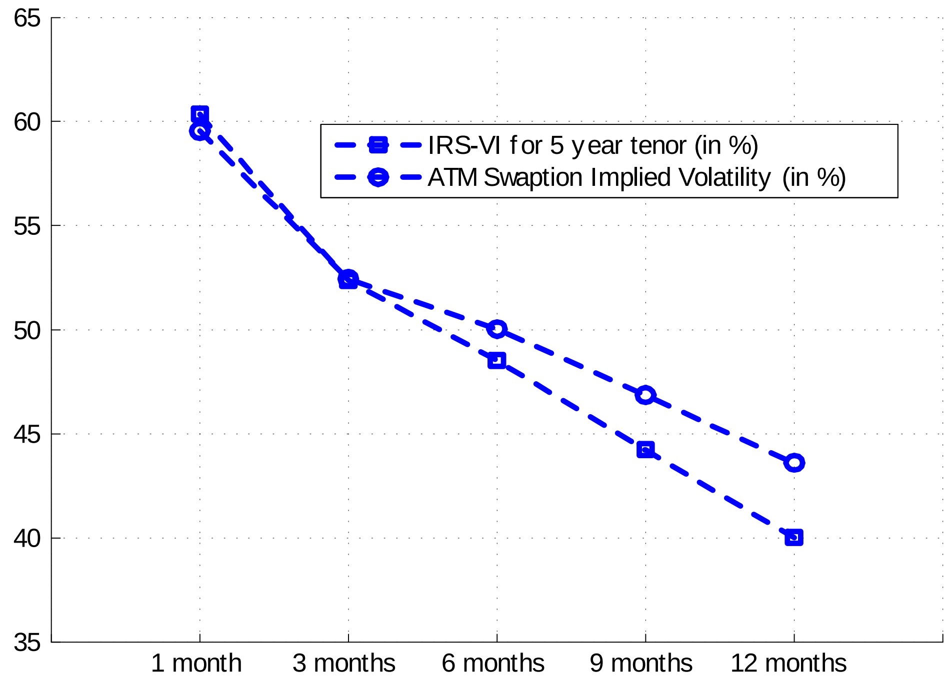

FIGURE 11. Term structure of the Interest Rate Swap Volatility Index (IRS-VI)—February 13, 2009.

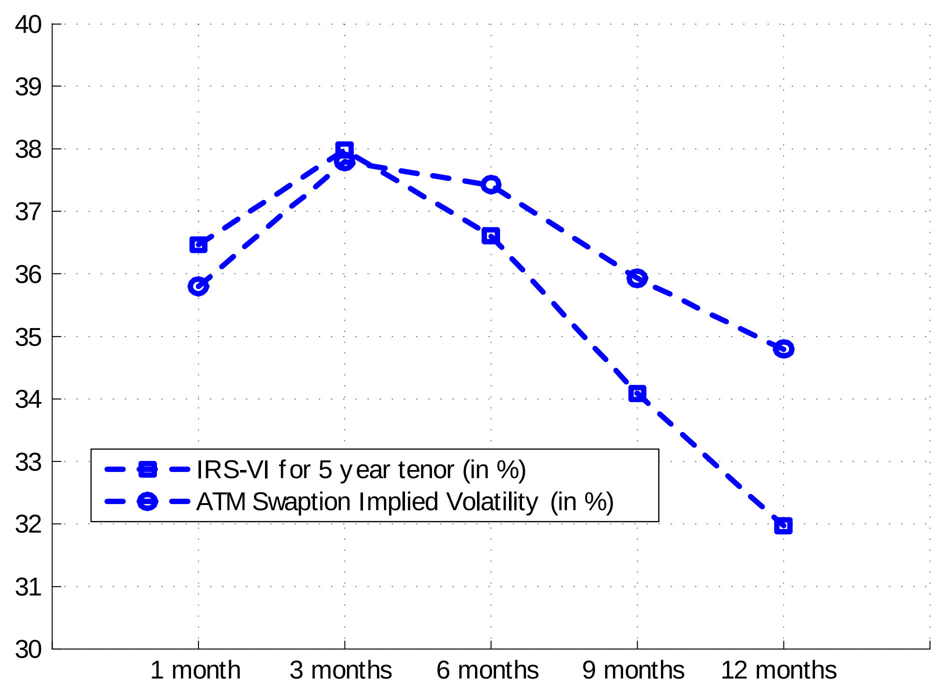

FIGURE 12. Term structure of the Interest Rate Swap Volatility Index (IRS-VI)—February 12, 2010.

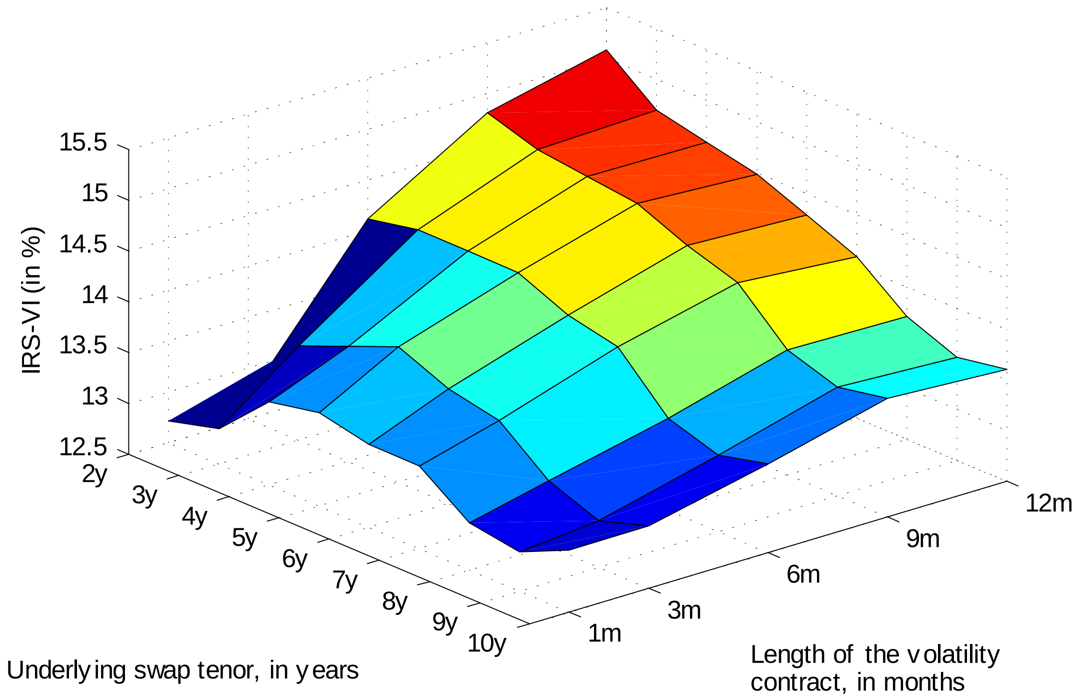

FIGURE 13. Interest Rate Swap Volatility Index (IRS-VI)—February 16, 2007.

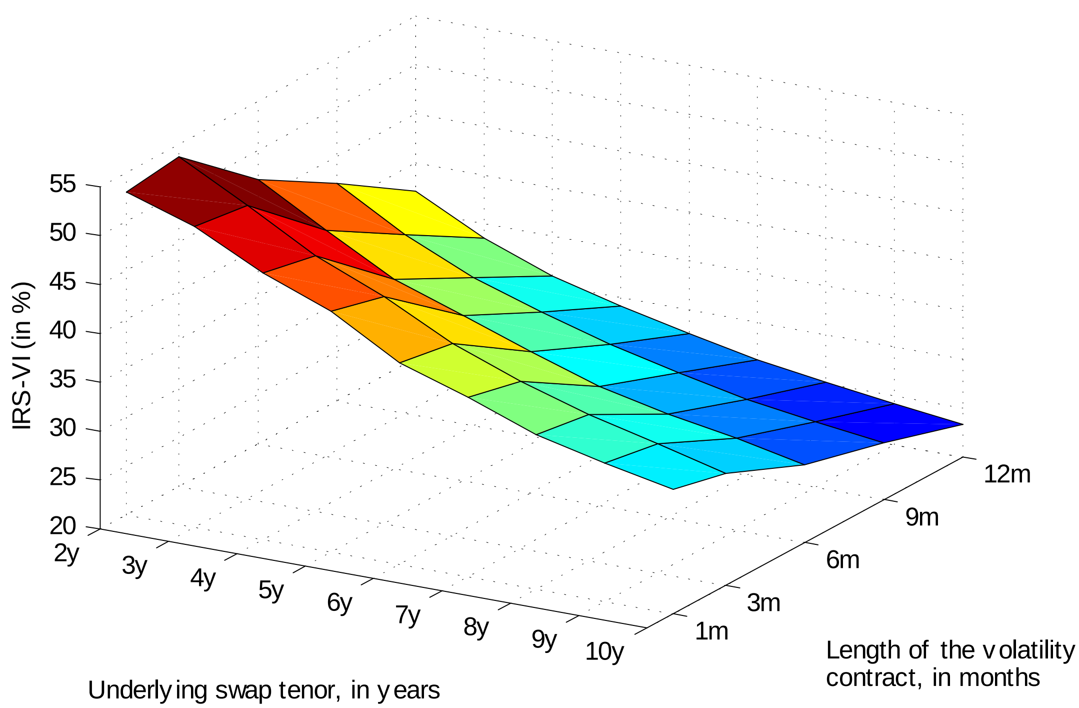

FIGURE 14. Interest Rate Swap Volatility Index (IRS-VI)—February 15, 2008.

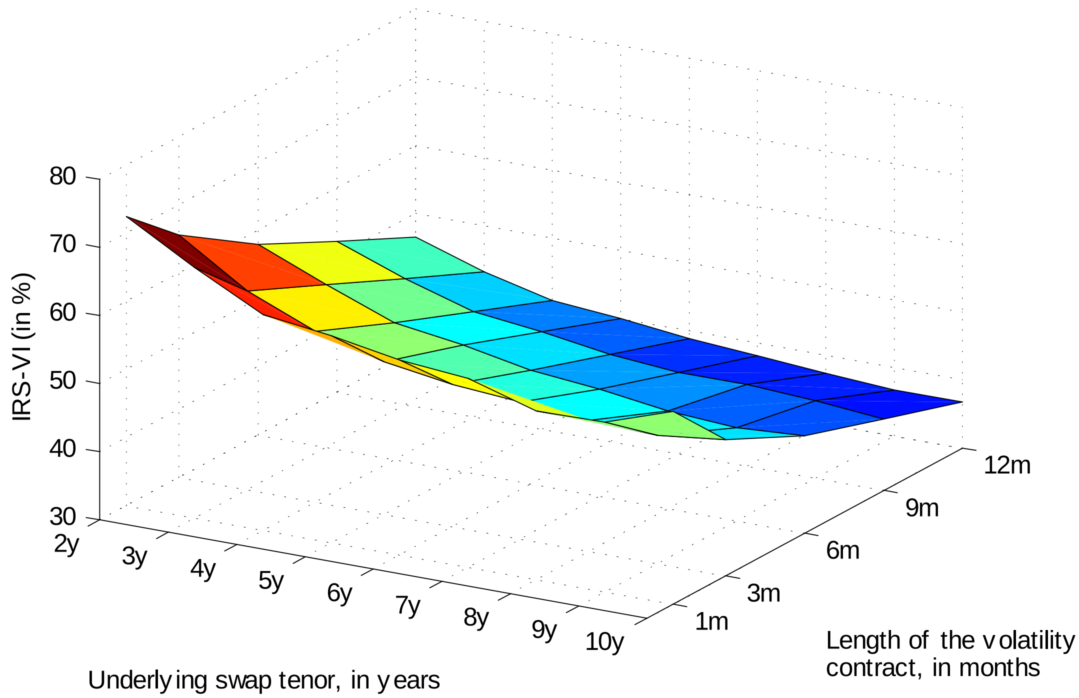

FIGURE 15. Interest Rate Swap Volatility Index (IRS-VI)—February 13, 2009.

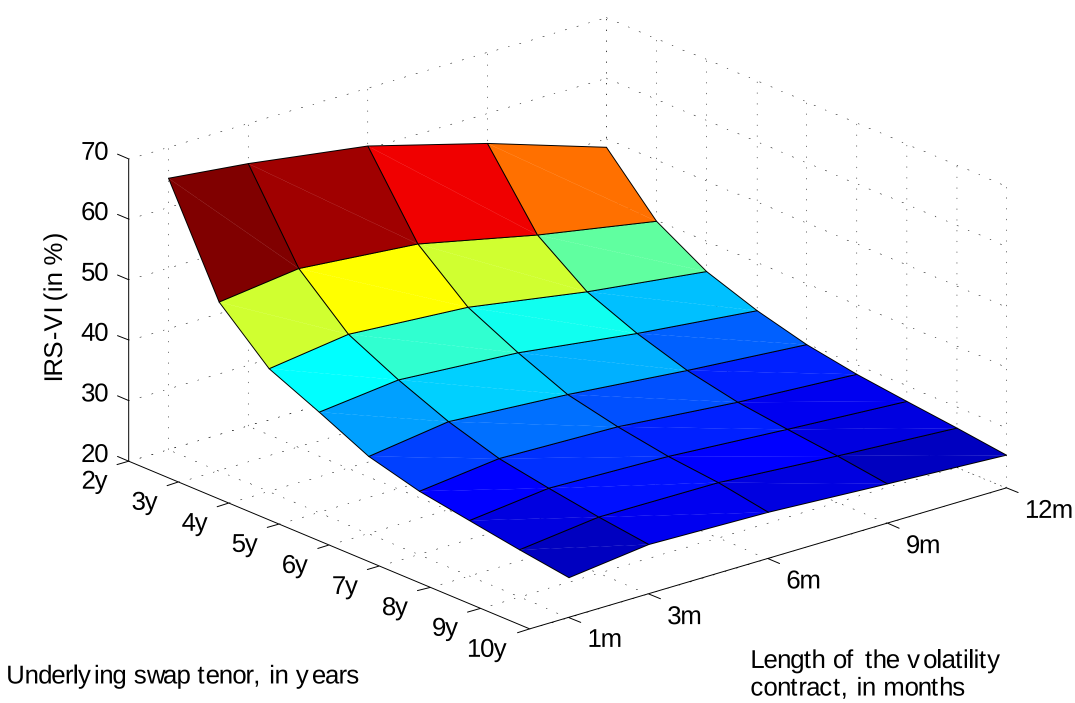

FIGURE 16. Interest Rate Swap Volatility Index (IRS-VI)—February 12, 2010.

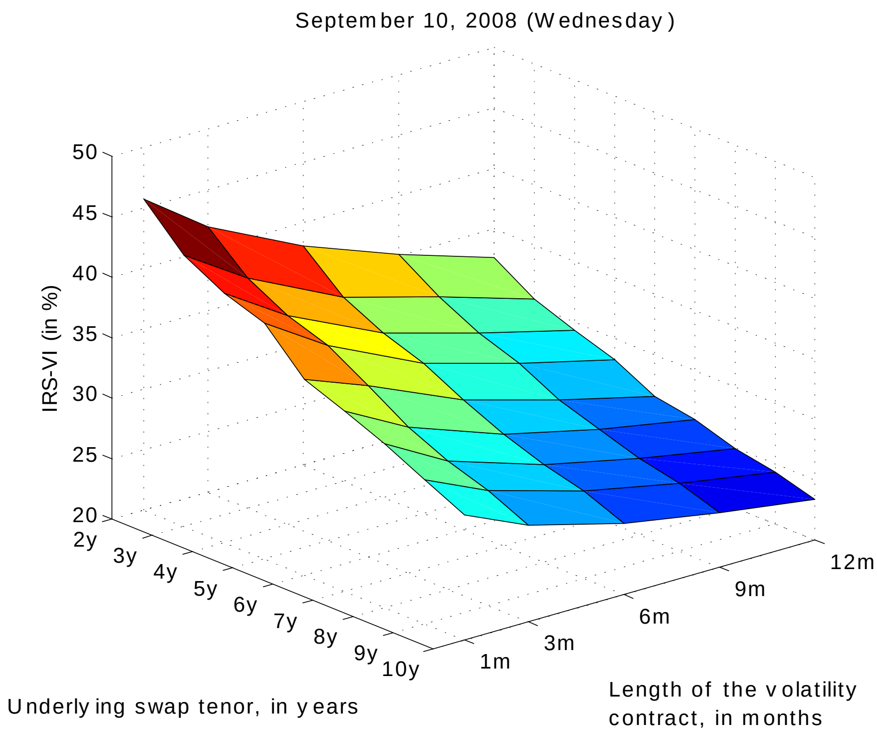

FIGURE 17. Interest Rate Swap Volatility Index (IRS-VI) around the Lehman's collapse days: I.

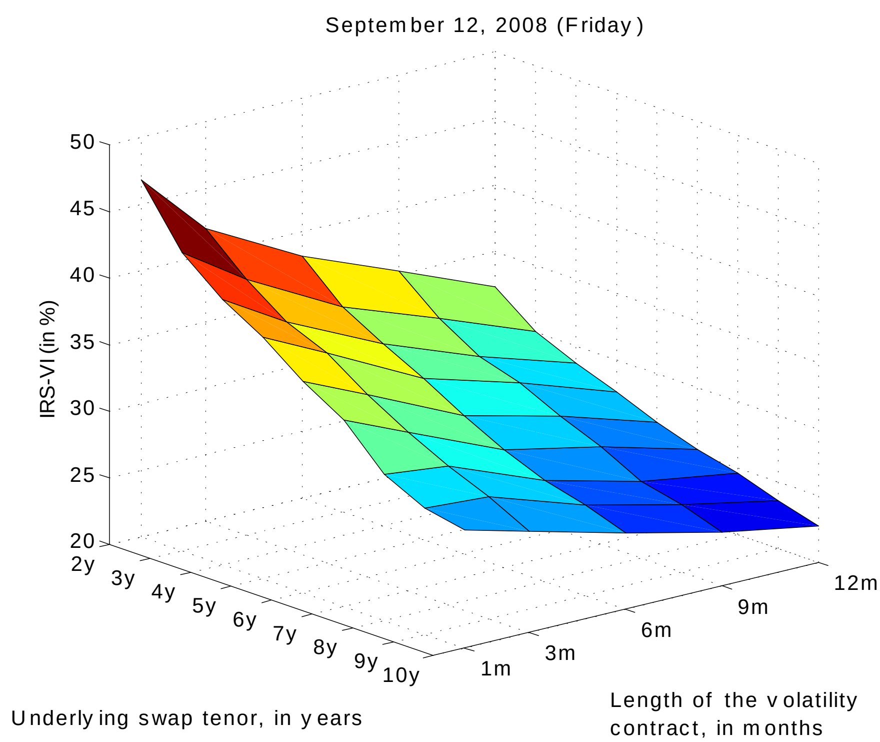

FIGURE 18. Interest Rate Swap Volatility Index (IRS-VI) around the Lehman's collapse days: II.

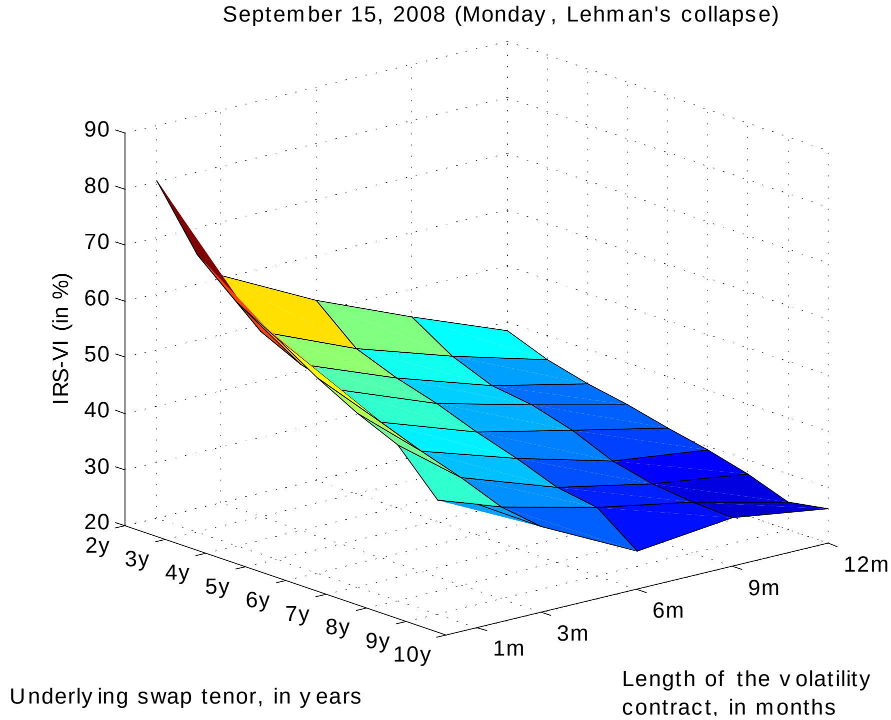

FIGURE 19. Interest Rate Swap Volatility Index (IRS-VI) around the Lehman's collapse days: III.

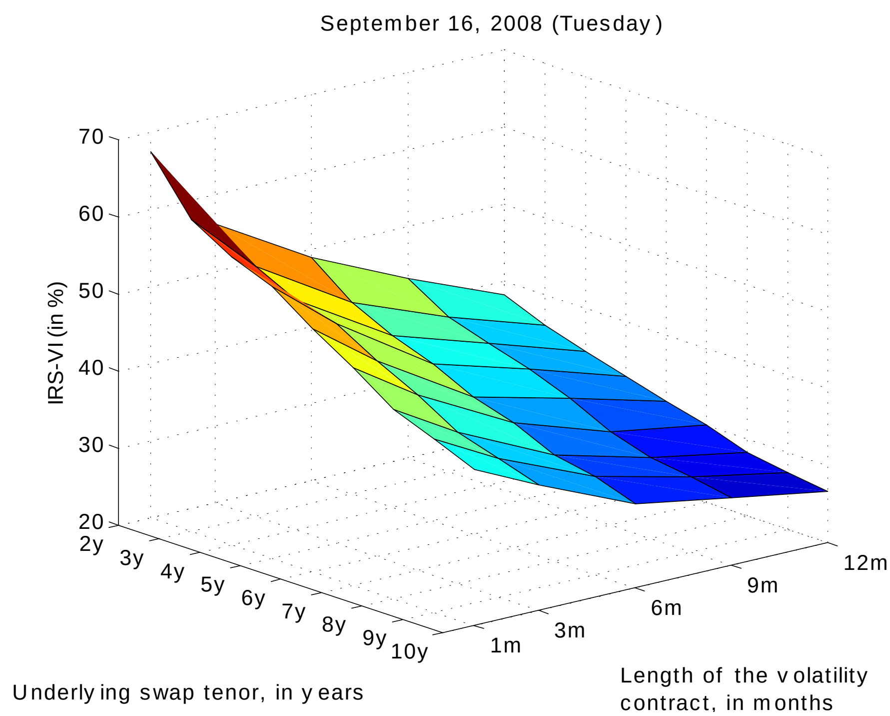

FIGURE 20. Interest Rate Swap Volatility Index (IRS-VI) around the Lehman's collapse days: IV.

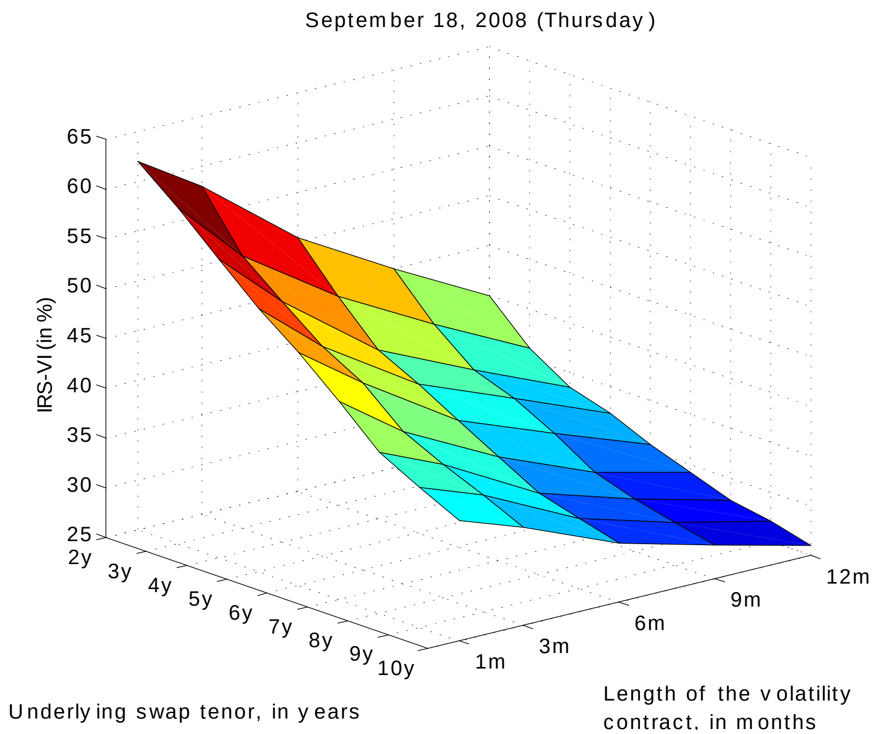

FIGURE 21. Interest Rate Swap Volatility Index (IRS-VI) around the Lehman's collapse days: V.

Technical Appendix

A. Notation, assumptions and preliminary facts

- Let $P_t(T)$ be the price as of time t of a zero coupon bond expiring at time $T > t$.

- The forward swap rate as of time t is the fixed interest rate that makes the value of a forward starting swap equal to zero. For a forward start swap contract with future reset dates $T_0, \cdots, T_{n-1}$ and payment periods $T_1 - T_0, \cdots, T_n - T_{n-1}$, it is defined as:

$$R_t(T_1, \cdots, T_n) = \frac{P_t(T_0) - P_t(T_n)}{\mathrm{PVBP}_t(T_1, \cdots, T_n)}, \tag{A.1}$$

  where $\mathrm{PVBP}_t(T_1, \cdots, T_n)$ is the "price value of the basis point" of the swap,

$$\mathrm{PVBP}_t(T_1, \cdots, T_n) = \sum_{i=1}^{n} \delta_{i-1} P_t(T_i), \tag{A.2}$$

  and $\delta_i = T_{i+1} - T_i$ are the lengths of the reset intervals.

- Let r_s be the instantaneous interest rate as of time s, and define the value of the money market account as of time $\tau \geq t$ (MMA, for short) as $M_\tau = e^{\int_t^\tau r_s ds}$.

- Let Q be the risk-neutral probability, under which all traded assets discounted by the MMA are martingales. Define the Radon-Nikodym derivative of Q_{swap} with respect to Q as:

$$\left.\frac{dQ_{\text{swap}}}{dQ}\right|_{\mathbb{F}_T} = e^{-\int_t^T r_s ds} \frac{\mathrm{PVBP}_T(T_1, \cdots, T_n)}{\mathrm{PVBP}_t(T_1, \cdots, T_n)}, \tag{A.3}$$

  where $\mathbb{F}_T$ is the information set as of time T, $\mathbb{E}_Q[\cdot]$ denotes the expectation taken under the risk-neutral probability, and $\mathbb{E}_{Q_{\text{swap}}}[\cdot]$ is the expectation taken under the Q_{swap} probability.

- The forward swap rate is solution to (see, e.g., Mele, 2011, Chapter 12):

$$\frac{dR_s(T_1, \cdots, T_n)}{R_s(T_1, \cdots, T_n)} = \sigma_s(T_1, \cdots, T_n)\, dW_s^*, \quad s \in [t, T], \tag{A.4}$$

  where W_t^* is a multidimensional Wiener process under the Q_{swap} probability, and $\sigma_t(T_1, \cdots, T_n)$ is adapted to W_t^*.

- The payoff of the IRV forward agreement is:

$$V_n(t, T) \times \mathrm{PVBP}_T(T_1, \cdots, T_n), \tag{A.5}$$

  where $V_n(t, T)$ is the realized variance of the forward swap rate logarithmic changes in the interval $[t, T]$,

$$V_n(t, T) = \int_t^T \sigma_s^2(T_1, \cdots, T_n)\, ds, \tag{A.6}$$

  where we use the simplified notation,

$$\sigma_t^2(T_1, \cdots, T_n) \equiv \|\sigma_t(T_1, \cdots, T_n)\|^2 .$$

- In the absence of arbitrage, the value of a forward starting swap payer with fixed interest K is (see, e.g., Mele, 2011, Chapter 12),

$$\mathrm{SWAP}_t\left(K;T_1,\cdots,T_n\right)=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\left[R_t\left(T_1,\cdots,T_n\right)-K\right], \tag{A.7}$$

  and the prices of a payer and receiver swaptions expiring at T are, respectively,

$$\mathrm{SWPN}_t^{\mathrm{P}}\left(K,T;T_1,\cdots,T_n\right)=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\mathbb{E}_{Q_{\mathrm{swap}}}\left[R_T\left(T_1,\cdots,T_n\right)-K\right]^+, \tag{A.8}$$

  and

$$\mathrm{SWPN}_t^{\mathrm{R}}\left(K,T;T_1,\cdots,T_n\right)=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\mathbb{E}_{Q_{\mathrm{swap}}}\left[K-R_T\left(T_1,\cdots,T_n\right)\right]^+. \tag{A.9}$$

  The parity for payer and receiver swaptions is:

$$\mathrm{SWPN}_t^{\mathrm{P}}\left(K,T;T_1,\cdots,T_n\right)=\mathrm{SWAP}_t\left(K;T_1,\cdots,T_n\right)+\mathrm{SWPN}_t^{\mathrm{R}}\left(K,T;T_1,\cdots,T_n\right). \tag{A.10}$$

B. P&L of option-based volatility trading

B.1. Definitions

The price of a payer swaption in Eq. (A.8) is known in closed-form, once we assume that the volatility σ_t in Eq. (A.4) and, hence, the integrated variance $V_n\left(t,T\right)$ in Eq. (A.6), are deterministic. It is given by Black's (1976) formula:

$$\mathrm{SWPN\_Bl}_t^{\mathrm{P}}\left(R_t^n,\mathrm{PVBP}_t,K,T;\bar{V}\right)=\mathrm{PVBP}_t\cdot Z_t\left(R_t^n,T,K;\bar{V}\right), \tag{B.1}$$

where:

$$Z_t(R_t;T,K;\bar{V})=R_t\Phi\left(d_t\right)-K\Phi(d_t-\sqrt{\bar{V}}),\quad d_t=\frac{\ln\frac{R_t}{K}+\frac{1}{2}\bar{V}}{\sqrt{\bar{V}}},$$

Φ denotes the cumulative standard normal distribution, and to alleviate notation, $R_t^n\equiv R_t\left(T_1,\cdots,T_n\right)$, $\mathrm{PVBP}_t\equiv\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)$, and $\bar{V}$ is the constant value of $V_n\left(t,T\right)$ in Eq. (A.6). By the definition of the forward swap rate in Eq. (A.1), we have:

$$\mathrm{SWPN\_Bl}_t^{\mathrm{P}}\left(R_t^n,\mathrm{PVBP}_t,K,T;\bar{V}\right)\equiv\left(P_t\left(T\right)-P_t\left(T_n\right)\right)\Phi\left(d_t\right)-\mathrm{PVBP}_tK\Phi(d_t-\sqrt{\bar{V}}). \tag{B.2}$$

Eq. (B.2) shows the swaption can be hedged through portfolios of zero coupon bonds: (i) long $\Phi\left(d_t\right)$ units of a portfolio which is long one zero expiring at T and short one zero expiring at T_n, and (ii) short $K\Phi\left(d_t-s\right)$ units of the bonds basket PVBP_t. Alternatively, note that:

$$\mathrm{SWPN\_Bl}_t^{\mathrm{P}}\left(R_t^n,\mathrm{PVBP}_t,K,T;\bar{V}\right)=\mathrm{SWAP}_t\left(K;T_1,\cdots,T_n\right)\Phi\left(d_t\right)+\mathrm{PVBP}_tK\left(\Phi\left(d_t\right)-\Phi(d_t-\sqrt{\bar{V}})\right). \tag{B.3}$$

Eq. (B.3) shows the swaption can equally be hedged as follows: (i) long $\Phi\left(d_t\right)$ units of a swap, and (ii) long $K(\Phi\left(d_t\right)-\Phi(d_t-\sqrt{\bar{V}}))$ units of the bonds basket PVBP_t.

Next, and against the assumption underlying Eq. (B.2) and Eq. (B.3), we assume the forward swap rate has stochastic volatility, σ_t say, as in Eq. (A.4), which it does, for example, in the models of Appendix F. We wish to have a view that volatility will raise, say, compared to the current implied volatility, defined as the value,

$$\mathrm{IV}_t=\sqrt{\frac{\bar{V}}{T-t}}, \tag{B.4}$$

such that, once re-normalized again by $T-t$ and, then, inserted into Eq. (B.2), it delivers the swaption market price.

B.2. Delta-hedged swaptions strategies

Consider, first, the strategy to purchase the swaption, and hedge it using one of the portfolios underlying either Eq. (B.2) or Eq. (B.3). We derive the P&L of this strategy assuming the hedging portfolio is that relying on Eq. (B.2). The value of the hedging strategy, υ say, satisfies:

$$\upsilon_\tau = a_\tau \left(P_\tau\left(T\right) - P_\tau\left(T_n\right)\right) + b_\tau \mathrm{PVBP}_\tau, \quad \tau \in \left[t, T\right],$$

where, denoting for simplicity $R_t \equiv R_t^n$,

$$\begin{aligned}
&\upsilon_t = \mathrm{SWPN_Bl}_t^{\mathrm{P}}\left(R_t, \mathrm{PVBP}_t, K, T, \left(T-t\right)\mathrm{IV}_t^2\right), \\
&a_\tau = \Phi\left(\frac{\ln\frac{R_\tau}{K} + \frac{1}{2}\left(T-t\right)\mathrm{IV}_t^2}{\sqrt{T-t}\mathrm{IV}_t}\right) \quad \text{and} \quad b_\tau = -K\Phi\left(\frac{\ln\frac{R_\tau}{K} - \frac{1}{2}\left(T-t\right)\mathrm{IV}_t^2}{\sqrt{T-t}\mathrm{IV}_t}\right).
\end{aligned} \tag{B.5}$$

Because the hedging strategy is self-financed, $d\upsilon_\tau = a_\tau d\left[P_\tau\left(T\right) - P_\tau\left(T_n\right)\right] + b_\tau d\mathrm{PVBP}_\tau$, and, hence:

$$d\upsilon_\tau = \left[\mu_\tau^b \upsilon_\tau + \left(\mu_\tau^{\Delta P} - \mu_\tau^b\right) a_\tau \left(P_\tau\left(T\right) - P_\tau\left(T_n\right)\right)\right] d\tau + \left[\sigma_\tau^b \upsilon_\tau + \left(\sigma_\tau^{\Delta P} - \sigma_\tau^b\right) a_\tau \left(P_\tau\left(T\right) - P_\tau\left(T_n\right)\right)\right] dW_\tau, \tag{B.6}$$

where W_τ is a Wiener process under the physical probability and, accordingly, μ_τ^b and σ_τ^b denote the drift and instantaneous volatility of $\frac{d\mathrm{PVBP}_\tau}{\mathrm{PVBP}_\tau}$, and $\mu_\tau^{\Delta P}$ and $\sigma_\tau^{\Delta P}$ denote the drift and instantaneous volatility of $\frac{d(P_\tau(T) - P_\tau(T_n))}{P_\tau(T) - P_\tau(T_n)}$.

On the other hand, consider the swaption price in Eq. (B.1) with $\bar{V}$ replaced by $\left(T-t\right)\mathrm{IV}_t^2$, as in Eq. (B.4), i.e. $\mathrm{SWPN\_Bl}_\tau^{\mathrm{P}} \equiv \mathrm{SWPN\_Bl}_\tau^{\mathrm{P}}\left(R_\tau, \mathrm{PVBP}_\tau, K, T, \left(T-t\right)\mathrm{IV}_t^2\right)$. By Itô's lemma, the definition of the forward swap rate in Eq. (A.1), and the partial differential equation satisfied by the pricing function $Z_t\left(R, T, K; \bar{V}\right)$, this price changes as follows:

$$\begin{aligned}
&d\mathrm{SWPN_Bl}_\tau^{\mathrm{P}} \\
&= \mathrm{PVBP}_\tau dZ_\tau + Z_\tau d\mathrm{PVBP}_\tau + dZ_\tau d\mathrm{PVBP}_\tau \\
&= \mathrm{PVBP}_\tau \Bigg(\underbrace{\frac{\partial Z_\tau}{\partial \tau} + \frac{1}{2}\frac{\partial^2 Z_\tau}{\partial R^2} R_\tau^2 \mathrm{IV}_t^2}_{=0}\Bigg) d\tau + \mathrm{PVBP}_\tau \left(\frac{1}{2}\frac{\partial^2 Z_\tau}{\partial R^2} R_\tau^2 \left(\sigma_\tau^2 - \mathrm{IV}_t^2\right) + \frac{\partial Z_\tau}{\partial R}\mu_\tau^R R_\tau\right) d\tau \\
&\quad + \left(\mu_\tau^b \mathrm{SWPN_Bl}_\tau^{\mathrm{P}} + \frac{\partial Z_\tau}{\partial R}\sigma_\tau^b \sigma_\tau \left(P_\tau\left(T\right) - P_\tau\left(T_n\right)\right)\right) d\tau \\
&\quad + \left(\frac{\partial Z_\tau}{\partial R}\sigma_\tau \left(P_\tau\left(T\right) - P_\tau\left(T_n\right)\right) + \sigma_\tau^b \mathrm{SWPN_Bl}_\tau^{\mathrm{P}}\right) dW_\tau,
\end{aligned} \tag{B.7}$$

where μ_τ^R is the drift of $\frac{dR_\tau}{R_\tau}$ under the physical probability, and equals

$$\mu_\tau^R = \mu_\tau^{\Delta p} - \mu_\tau^b - \sigma_\tau \sigma_\tau^b, \quad \sigma_\tau = \sigma_\tau^{\Delta p} - \sigma_\tau^b, \tag{B.8}$$

where the second relation follows by Itô's lemma. By using again the definition of the forward swap rate in Eq. (A.1), and then Eq. (B.8) and the relation $a_\tau = \frac{\partial Z_\tau}{\partial R}$, we can integrate the difference between $d\mathrm{SWPN\_Bl}_\tau^{\mathrm{P}}$ in Eq. (B.7), and $d\upsilon_\tau$ in Eq. (B.6), so as to obtain that the P&L at the swaption maturity is:

$$\begin{aligned}
\mathrm{SWPN_Bl}_T^{\mathrm{P}} - \upsilon_T &= \left[R_T - K\right]^+ - \upsilon_T \\
&= \frac{1}{2}\int_t^T \frac{\partial^2 Z_\tau}{\partial R^2} R_\tau^2 \left(\sigma_\tau^2 - \mathrm{IV}_t^2\right)\left(\Lambda_{\tau,T}\mathrm{PVBP}_\tau\right) d\tau + \int_t^T \Lambda_{\tau,T}\sigma_\tau^b \left(\mathrm{SWPN_Bl}_\tau^{\mathrm{P}} - \upsilon_\tau\right) dW_\tau,
\end{aligned} \tag{B.9}$$

where we have used the first relation in Eqs. (B.5), $\text{SWPN\_Bl}_t^{\text{P}} = \upsilon_t$, and defined $\Lambda_{\tau,T} = e^{\int_\tau^T \mu_s^b ds}$. The approximation in Eq. (2) relies on: (i) $\text{PVBP}_T \approx \Lambda_{\tau,T}\text{PVBP}_\tau$; (ii) $\widetilde{\frac{\Delta R_t}{R_t}} \approx dW_t$; and (iii) disregarding the term $\Lambda_{\tau,T}$ inside the stochastic integral in Eq. (B.9), which we merely made to simplify the presentation.

B.3. Straddles strategies

Finally, we consider the P&L relating to trading the straddle. The value of the straddle is, $\text{STRADDLE}_\tau = \text{SWPN}_\tau^{\text{P}} + \text{SWPN}_\tau^{\text{R}}$. By Black's (1976) formula, the price of the receiver swaption is:

$$\begin{gathered}\text{SWPN\_Bl}_t^{\text{R}}\left(R_t^n, \text{PVBP}_t, K, T; \bar{V}\right) = \text{PVBP}_t \cdot \hat{Z}_t\left(R_t^n, T, K; \bar{V}\right), \\ \hat{Z}_t\left(R_t, T, K; \bar{V}\right) = K\left(1 - \Phi(d_t - \sqrt{\bar{V}})\right) - R_t\left(1 - \Phi(d_t)\right), \quad d_t = \frac{\ln\frac{R_t}{K} + \frac{1}{2}\bar{V}}{\sqrt{\bar{V}}}.\end{gathered} \tag{B.10}$$

Therefore, the dynamics of $\text{SWPN\_Bl}_\tau^{\text{R}}$ are the same as $\text{SWPN\_Bl}_\tau^{\text{P}}$ in Eq. (B.7), but with $\hat{Z}_\tau$ replacing Z_t. Moreover, we have, by Eq. (A.10) and Eq. (A.7),

$$\frac{\partial \text{STRADDLE}_\tau}{\partial R} = \text{PVBP}_\tau\left(1 + 2\frac{\partial \hat{Z}_\tau}{\partial R}\right), \quad \frac{\partial \hat{Z}_\tau}{\partial R} = \frac{\partial Z_\tau}{\partial R} - 1,$$

where Z_τ is as in Eq. (B.1). Assuming the straddle delta is sufficiently small, which by the previous equation it is when $2\frac{\partial Z_\tau}{\partial R} \approx 1$, the value of the straddle is, by Eq. (B.7), and the previous arguments about the dynamics of $\text{SWPN\_Bl}_\tau^{\text{R}}$:

$$\begin{aligned}&\text{STRADDLE}_T - \text{STRADDLE}_t \\ &= \int_t^T \frac{\partial^2 Z_\tau}{\partial R^2} R_\tau^2\left(\sigma_\tau^2 - \text{IV}_t^2\right)\left(\Lambda_{\tau,T}\text{PVBP}_\tau\right) d\tau + \int_t^T \Lambda_{\tau,T}\sigma_\tau^b \text{STRADDLE}_\tau dW_\tau.\end{aligned} \tag{B.11}$$

The approximation in Eq. (3) relies on the same arguments leading to Eq. (2). Note that the approximation $2\frac{\partial Z_\tau}{\partial R} \approx 1$ is quite inadequate as the forward swap rates drifts away from ATM—a very well-known feature discussed, e.g. by Mele (2011, Chapter 10) in the case of equity straddles.

C. Spanning the contracts

C.1. Pricing

We rely on spanning arguments similar to those utilized by Bakshi and Madan (2000) and Carr and Madan (2001) for equities. To alleviate notation, we set $R_t \equiv R_t^n = R_t(T_1, \cdots, T_n)$. By the usual Taylor's expansion with remainder,

$$\ln\frac{R_T}{R_t} = \frac{1}{R_t}\left(R_T - R_t\right) - \int_0^{R_t}\left(K - R_T\right)^+ \frac{1}{K^2}dK - \int_{R_t}^\infty \left(R_T - K\right)^+ \frac{1}{K^2}dK. \tag{C.1}$$

Multiplying both sides of the previous equation by $\text{PVBP}_t(T_1, \cdots, T_n)$, and taking expectations under Q_{swap}, defined through Eq. (A.3):

$$\text{PVBP}_t(T_1, \cdots, T_n)\,\mathbb{E}_{Q_{\text{swap}}}\left(\ln\frac{R_T}{R_t}\right) = -\int_0^{R_t}\frac{\text{SWPN}_t^{\text{R}}(K, T; T_n)}{K^2}dK - \int_{R_t}^\infty \frac{\text{SWPN}_t^{\text{P}}(K, T; T_n)}{K^2}dK, \tag{C.2}$$

where we have used (i) the fact that by Eq. (A.4), the forward swap rate is a martingale under Q_{swap}, and (ii) the pricing Equations (A.8) and (A.9).

By Eq. (A.4), and a change of measure obtained through Eq. (A.3),

$$-2\mathbb{E}_{Q_{\text{swap}}}\left(\ln\frac{R_T}{R_t}\right)=\mathbb{E}_{Q_{\text{swap}}}\left[\int_t^T\sigma_s^2\left(T_1,\cdots,T_n\right)ds\right]$$
$$=\frac{1}{\text{PVBP}_t\left(T_1,\cdots,T_n\right)}\mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(\text{PVBP}_T\left(T_1,\cdots,T_n\right)\int_t^T\sigma_s^2\left(T_1,\cdots,T_n\right)ds\right)\right]. \tag{C.3}$$

Combining this relation with Eq. (C.2) leaves:

$$\mathbb{F}_{\text{var},n}\left(t,T\right)=\mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(\text{PVBP}_T\left(T_1,\cdots,T_n\right)\int_t^T\sigma_s^2\left(T_1,\cdots,T_n\right)ds\right)\right]$$
$$=2\left[\int_0^{R_t}\frac{\text{SWPN}_t^{\text{R}}\left(K,T;T_1,\cdots,T_n\right)}{K^2}dK+\int_{R_t}^{\infty}\frac{\text{SWPN}_t^{\text{P}}\left(K,T;T_1,\cdots,T_n\right)}{K^2}dK\right]. \tag{C.4}$$

By Eqs. (A.5) and (A.6), the left hand side of the previous equation is the price of the IR forward variance agreement, and Eq. (8) is its approximation. The claim in the main text that if $V_n\left(t,T\right)$ and the path of the short-term rate in the time interval $[t,T]$ were independent, the index IRS-VI$_n\left(t,T\right)$ in Eq. (25) would be the risk-neutral expectation of $V_n\left(t,T\right)$, follows by Eq. (C.3) and the definition of the Radon-Nikodym derivative of Q_{swap} with respect to Q in Eq. (A.3).

To derive the IRV swap rate $\mathbb{P}_{\text{var},n}\left(t,T\right)$ in Eq. (9), note that by Eq. (6), it solves:

$$0=\mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(V_n\left(t,T\right)\times\text{PVBP}_T\left(T_1,\cdots,T_n\right)-\mathbb{P}_{\text{var},n}\left(t,T\right)\right)\right],$$

which, after rearranging terms, yields Eq. (9). As for the Standardized IRV swap rate $\mathbb{P}^*_{\text{var},n}\left(t,T\right)$ in Eq. (10), note that by Eq. (7), it satisfies:

$$0=\mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(V_n\left(t,T\right)-\mathbb{P}^*_{\text{var},n}\left(t,T\right)\right)\times\text{PVBP}_T\left(T_1,\cdots,T_n\right)\right],$$

which by the definition of the Radon-Nikodym derivative in Eq. (A.3), yields Eq. (10), after rearranging terms.

Finally, as claimed in the main text, Eq. (25) can be simplified, using the Black's (1976) formula. By replacing Eq. (8) into Eq. (25), and using the Black's formulae in Eqs. (B.1) and (B.10),

$$\text{IRS-VI}_n\left(t,T\right)$$
$$=\sqrt{\frac{2}{T-t}\left[\sum_{i:K_i<R_t}\frac{\hat{Z}_t\left(R_t^n,T,K_i;(T-t)\cdot\text{IV}_{i,t}^2\right)}{K_i^2}\Delta K_i+\sum_{i:K_i\geq R_t}\frac{Z_t\left(R_t^n,T,K_i;(T-t)\cdot\text{IV}_{i,t}^2\right)}{K_i^2}\Delta K_i\right]}, \tag{C.5}$$

where the expressions for Z_t and $\hat{Z}_t$ are given in Eqs. (B.1) and (B.10), R_t^n is the current forward swap rate for maturity T and tenor length T_n-T, $R_t^n\equiv R_t\left(T_1,\cdots,T_n\right)$, and, finally, IV$_{i,t}$ denotes the time t implied percentage volatility for swaptions with strike equal to K_i.

C.2. Marking to market

We first derive Eq. (19). For a given $\tau\in(t,T)$, we need to derive the following conditional expectation of the payoff Var-Swap$_n\left(t,T\right)$ in Eq. (6):

$$\mathbb{E}_Q^{\tau}\left[e^{-\int_\tau^T r_u du}\text{Var-Swap}_n\left(t,T\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}_Q^{\tau}\left[e^{-\int_{\tau}^{T} r_u du}\left(V_n(t,\tau)+V_n(\tau,T)\right)\times \text{PVBP}_T(T_1,\cdots,T_n)\right] - P_\tau(T)\,\mathbb{P}_{\text{var},n}(t,T) \\
&= V_n(t,\tau)\,\text{PVBP}_\tau(T_1,\cdots,T_n) + \mathbb{F}_{\text{var},n}(\tau,T) - \frac{P_\tau(T)}{P_t(T)}\mathbb{F}_{\text{var},n}(t,T),
\end{aligned} \tag{C.6}
$$

where $\mathbb{E}_Q^{\tau}$ denotes expectation under Q, conditional upon all information up to time τ, and we have used the definition of the Radon-Nikodym derivative of Q_{swap} in Eq. (A.3) and expression for $\mathbb{F}_{\text{var},n}(\cdot,T)$ in Eq. (C.4). Eq. (19) follows after plugging the expression for $\mathbb{P}_{\text{var},n}(t,T)$ in Eq. (9) into Eq. (C.6).

Next, we derive Eq. (20). We have, utilizing the expression of Var-Swap$_n^*(t,T)$ in Eq. (7),

$$
\begin{aligned}
&\mathbb{E}_Q^{\tau}\left[e^{-\int_{\tau}^{T} r_u du}\text{Var-Swap}_n^*(t,T)\right] \\
&= \mathbb{E}_Q^{\tau}\left[e^{-\int_{\tau}^{T} r_u du}\left[V_n(t,\tau)+V_n(\tau,T)-\mathbb{P}_{\text{var},n}^*(t,T)\right]\times \text{PVBP}_T(T_1,\cdots,T_n)\right] \\
&= \text{PVBP}_\tau(T_1,\cdots,T_n)\left[V_n(t,\tau)+\mathbb{P}_{\text{var},n}^*(\tau,T)-\mathbb{P}_{\text{var},n}^*(t,T)\right],
\end{aligned}
$$

where the second equality follows by the expression for $\mathbb{F}_{\text{var},n}(\cdot,T)$ in Eq. (C.4), and by Eq. (10).

C.3. Hedging

We provide the details leading to Table I, as those for Table II are nearly identical. By Itô's lemma:

$$
\begin{aligned}
&\text{PVBP}_T(T_1,\cdots,T_n)\int_t^T \sigma_s^2(T_1,\cdots,T_n)\,ds \\
&= 2\text{PVBP}_T(T_1,\cdots,T_n)\int_t^T \frac{dR_s(T_1,\cdots,T_n)}{R_s(T_1,\cdots,T_n)} - 2\text{PVBP}_T(T_1,\cdots,T_n)\ln\frac{R_T(T_1,\cdots,T_n)}{R_t(T_1,\cdots,T_n)},
\end{aligned} \tag{C.7}
$$

where, by Eq. (C.1), the expression for the swap value in Eq. (1) and that for the swaption premium in Eqs. (A.9)-(A.10), the second term on the right hand side is the payoff at T of a portfolio set up at t, which is two times: (a) short $1/R_t(T_1,\cdots,T_n)$ units a fixed interest payer swap struck at $R_t(T_1,\cdots,T_n)$, and (b) long a continuum of out-of-the-money swaptions with weights $K^{-2}dK$. This portfolio is the static position (ii) in Table I of the main text. By Eqs. (C.1), (C.2) and (C.4), its cost is $\mathbb{F}_{\text{var},n}(t,T)$. We borrow $\mathbb{F}_{\text{var},n}(t,T)$ at time t, and repay it back at time T, as in row (iii) of Table I of the main text.

Next, we derive the self-financing portfolio of zero coupon bonds (i) in Table I, by designing it to be worthless at time t, and to replicate the first term on the right hand side of Eq. (C.7). First, note that by Eq. (A.1), the forward swap rate as of time s can be replicated through a portfolio that is long zeros maturing at T_0 and short zeros maturing at T_n, with equal weights $1/\text{PVBP}_s(T_1,\cdots,T_n)$. Consider, then, a self-financed strategy investing in (a) this portfolio, which we call the "forward swap rate" for simplicity, and (b) a MMA, as defined in Appendix A, with total value equal to:

$$
\upsilon_\tau = \theta_\tau R_\tau(T_1,\cdots,T_n) + \psi_\tau M_\tau,
$$

where θ_τ are the units in the forward swap rate and ψ_τ are the units in the MMA. Consider the following portfolio:

$$
\hat{\theta}_\tau R_\tau(T_1,\cdots,T_n) = \text{PVBP}_\tau(T_1,\cdots,T_n), \quad \hat{\psi}_\tau M_\tau = \text{PVBP}_\tau(T_1,\cdots,T_n)\left(\int_t^\tau \frac{dR_s(T_1,\cdots,T_n)}{R_s(T_1,\cdots,T_n)} - 1\right). \tag{C.8}
$$

We have that $\hat{\upsilon}_\tau = \hat{\theta}_\tau R_\tau + \hat{\psi}_\tau M_\tau$ satisfies:

$$
\hat{\upsilon}_\tau = \text{PVBP}_\tau(T_1,\cdots,T_n)\int_t^\tau \frac{dR_s(T_1,\cdots,T_n)}{R_s(T_1,\cdots,T_n)}, \tag{C.9}
$$

such that

$$\hat{v}_t = 0, \quad \text{and} \quad \hat{v}_T = \text{PVBP}_T\left(T_1, \cdots, T_n\right) \int_t^T \frac{dR_s\left(T_1, \cdots, T_n\right)}{R_s\left(T_1, \cdots, T_n\right)}.$$

Therefore, by going long two portfolios $(\hat{\theta}_\tau, \hat{\psi}_\tau)$, the first term on the right hand side and, then, by the previous results, the whole of the right hand side of Eq. (C.7), can be replicated and, so, hedged, provided $(\hat{\theta}_\tau, \hat{\psi}_\tau)$ is self-financed. We are only left to show $(\hat{\theta}_\tau, \hat{\psi}_\tau)$ is self-financed. We have:

$$\begin{aligned}
& d\hat{v}_\tau \\
& = \hat{\theta}_\tau R_\tau\left(T_1, \cdots, T_n\right) \frac{dR_\tau\left(T_1, \cdots, T_n\right)}{R_\tau\left(T_1, \cdots, T_n\right)} + \hat{\psi}_\tau M_\tau \frac{dM_\tau}{M_\tau} \\
& = \text{PVBP}_\tau\left(T_1, \cdots, T_n\right)\left(\frac{dR_\tau\left(T_1, \cdots, T_n\right)}{R_\tau\left(T_1, \cdots, T_n\right)} - r_\tau d\tau\right) + \left(\text{PVBP}_\tau\left(T_1, \cdots, T_n\right) \int_t^\tau \frac{dR_s\left(T_1, \cdots, T_n\right)}{R_s\left(T_1, \cdots, T_n\right)}\right) r_\tau d\tau \\
& = \text{PVBP}_\tau\left(T_1, \cdots, T_n\right)\left(\frac{dR_\tau\left(T_1, \cdots, T_n\right)}{R_\tau\left(T_1, \cdots, T_n\right)} - r_\tau d\tau\right) + r_\tau \hat{v}_\tau d\tau \\
& = \hat{\theta}_\tau R_\tau\left(\frac{dR_\tau\left(T_1, \cdots, T_n\right)}{R_\tau\left(T_1, \cdots, T_n\right)} - r_\tau d\tau\right) + r_\tau \hat{v}_\tau d\tau,
\end{aligned} \tag{C.10}$$

where the second line follows by the portfolio expressions for $(\hat{\theta}_\tau, \hat{\psi}_\tau)$ in Eq. (C.8), the third line holds by Eq. (C.9), and the fourth follows, again, by the expression for $\hat{\theta}_\tau R_\tau$ in Eq. (C.8). It is easy to check that the dynamics of $\hat{v}_\tau$ in Eq. (C.10) are those of a self-financed strategy.

D. Directional volatility trades

The P&L displayed in Figure 4 of Section 3 are calculated using daily data from January 1998 to December 2009, and comprise yield curve data as well as implied volatilities for swaptions. Yield curve data are needed to compute forward swap rates for a fixed 5 year tenor and the related realized volatilities, and implied volatilities are needed to compute P&Ls, as explained next.

As for the straddle, the strategy is to go long an at-the-money swaption straddle. Let t denote the beginning of the holding period (one month or three months). The terminal P&L of the straddle, that of as of time T say, is:

$$\text{PVBP}_T\left(T_1, \cdots, T_n\right)\left(\left[R_T\left(T_1, \cdots, T_n\right) - K\right]^+ + \left[K - R_T\left(T_1, \cdots, T_n\right)\right]^+\right) - \text{Straddle}_t\left(T_1, \cdots, T_n\right),$$

where $\text{Straddle}_t\left(T_1, \cdots, T_n\right)$ is the cost of the straddle at t, and $T - t$ is either one month (as in the top panel of Figure 4) or three months (as in the bottom panel of Figure 4).

Instead, the terminal P&L of the volatility swap contract is calculated consistently with Definition II and Eq. (9), as:

$$\frac{1}{T-t}\left(\text{PVBP}_T\left(T_1, \cdots, T_n\right) V_n\left(t, T\right) - \frac{\mathbb{F}^*_{\text{var},n}\left(t, T\right)}{P_t\left(T\right)}\right),$$

where $\mathbb{F}^*_{\text{var},n}\left(t, T\right)$ approximates the unnormalized strike of the contract to be entered at t. Its exact value, $\mathbb{F}_{\text{var},n}\left(t, T\right)$, is that in Eq. (8). By Eq. (25),

$$\frac{1}{T-t}\mathbb{F}_{\text{var},n}\left(t, T\right) = \text{PVBP}_t\left(T_1, \cdots, T_n\right) \cdot \text{IRS-VI}_n\left(t, T\right)^2,$$

which we approximate through:

$$\frac{1}{T-t}\mathbb{F}^*_{\text{var},n}\left(t, T\right) \equiv \text{PVBP}_t\left(T_1, \cdots, T_n\right) \cdot \text{ATM}_n\left(t, T\right)^2,$$

where $\text{ATM}_n\left(t, T\right)$ is the at-the-money implied volatility.

Finally, the realized variance rescaled by $T-t$, $\frac{1}{T-t}V_n(t,T)$, is calculated as:

$$\overline{\mathrm{RV}}_t^m = \left(\frac{21\cdot m}{252}\right)^{-1}\mathrm{RV}_t^m, \quad \text{where} \quad \mathrm{RV}_t^m = \sum_{i=t-21\cdot m}^{t}\left(\ln\frac{R_i(T_1,\cdots,T_n)}{R_{i-1}(T_1,\cdots,T_n)}\right)^2,$$

with $m\in\{1,3\}$, $T_1-t=\frac{1}{4}$ and $T_n-t=5$. The variance risk-premium is defined as $\overline{\mathrm{RV}}_t^m - \mathrm{ATM}_n^2(t-21\cdot m,T)$, where $T-t$ is either $\frac{1}{12}$ (for one month) or $\frac{3}{12}$ (for three months). Finally, to make the P&Ls of the volatility swap contract and the straddle line up to the same order of magnitude, the pictures in Figure 4 are obtained by re-multiplying the P&Ls of the volatility swap contracts by $\frac{1}{12}$ (top panel) and $\frac{3}{12}$ (bottom panel).

E. Local volatility surfaces

We develop arguments that hinge upon those Dumas (1995) and Britten-Jones and Neuberger (2000) put forth in the equity case. Consider the price of the payer swaption in Eq. (A.8), $\mathrm{SWPN}_t^{\mathrm{P}}(K,T;T_n)$. For simplicity, let $R_t\equiv R_t(T_1,\cdots,T_n)$. We have:

$$\frac{\partial\mathrm{SWPN}_t^{\mathrm{P}}}{\partial T} = \frac{\partial\ln\mathrm{PVBP}_t}{\partial T}\cdot\mathrm{SWPN}_t^{\mathrm{P}} + \mathrm{PVBP}_t\cdot\frac{d\mathbb{E}_{Q_{\mathrm{swap}}}(R_T-K)^+}{dT}. \tag{E.1}$$

Since the forward swap rate is a martingale under Q_{swap}, with volatility as in Eq. (A.4), we have:

$$\frac{d\mathbb{E}_{Q_{\mathrm{swap}}}(R_T-K)^+}{dT} = \frac{1}{2}\mathbb{E}_{Q_{\mathrm{swap}}}\left[\delta(R_T-K)R_T^2\sigma_T^2\right], \tag{E.2}$$

where $\delta(\cdot)$ is the Dirac's delta. We can elaborate the right hand side of Eq. (E.2), obtaining:

$$\begin{aligned}
\mathbb{E}_{Q_{\mathrm{swap}}}\left[\delta(R_T-K)R_T^2\sigma_T^2\right] &= \iint \delta(R_T-K)R_T^2\sigma_T^2\underbrace{\phi_T^{\mathrm{c}}(\sigma_T|R_T)\,\phi_T^{\mathrm{m}}(R_T)}_{\equiv\text{ joint density of }(\sigma_T,R_T)}dR_T d\sigma_T \\
&= K^2\phi_T^{\mathrm{m}}(K)\,\mathbb{E}_{Q_{\mathrm{swap}}}\left[\sigma_T^2\middle| R_T=K\right] \\
&= K^2\frac{\frac{\partial^2\mathrm{SWPN}_t^{\mathrm{P}}}{\partial K^2}}{\mathrm{PVBP}_t}\mathbb{E}_{Q_{\mathrm{swap}}}\left[\sigma_T^2\middle| R_T=K\right],
\end{aligned}$$

where $\phi_T^{\mathrm{c}}(\sigma_T|R_T)$ denotes the conditional density of σ_T given R_T under Q_{swap}, $\phi_T^{\mathrm{m}}(R_T)$ denotes the marginal density of R_T under Q_{swap}, and the third line follows by a well-known property of option-like prices. By replacing this result into Eq. (E.2) and then into Eq. (E.1), we obtain:

$$\frac{\partial\mathrm{SWPN}_t^{\mathrm{P}}}{\partial T} = \frac{\partial\ln\mathrm{PVBP}_t}{\partial T}\cdot\mathrm{SWPN}_t^{\mathrm{P}} + \frac{1}{2}K^2\frac{\partial^2\mathrm{SWPN}_t^{\mathrm{P}}}{\partial K^2}\mathbb{E}_{Q_{\mathrm{swap}}}\left[\sigma_T^2\middle| R_T=K\right].$$

Rearranging this equation, we obtain:

$$\sigma_{\mathrm{loc}}^2(K,T)\equiv\mathbb{E}_{Q_{\mathrm{swap}}}\left[\sigma_T^2\middle| R_T=K\right] = 2\frac{\frac{\partial\mathrm{SWPN}_t^{\mathrm{P}}}{\partial T}-\frac{\partial\ln\mathrm{PVBP}_t}{\partial T}\mathrm{SWPN}_t^{\mathrm{P}}}{K^2\frac{\partial^2\mathrm{SWPN}_t^{\mathrm{P}}}{\partial K^2}}. \tag{E.3}$$

Next, let the volatility of the forward swap rate in Eq. (A.4) be given by:

$$\sigma_s = \sigma(R_s,s)\,v_s,$$

where v_s is another random process. Define:

$$\sigma(R,s) = \frac{\sigma_{\mathrm{loc}}(R,s)}{\sqrt{\mathbb{E}_{Q_{\mathrm{swap}}}\left[v_s^2\middle| R_s=R\right]}}, \tag{E.4}$$

where $\sigma_{\text{loc}}(R, s)$ is as in Eq. (E.3). Consider the following model of the forward swap rate, under the Q_{swap} probability,

$$\frac{dR_s}{R_s} = \sigma(R_s, s)\, v_s dW_s^*, \quad s \in [t, T],$$

where $\sigma(R_s, s)$ is as in Eq. (E.4). This model is, theoretically, capable to match the cross-section of swaptions without errors, and can be used to price all of the non-traded swaptions in Eq. (8), through Montecarlo integration.

F. The contract and index in the Vasicek market

We consider two alternative models. In the first model, Vasicek's (1977), the short-term rate follows a Gaussian process:

$$dr_t = \phi(\bar{r} - r_t)\, dt + \sigma_v d\hat{W}_t, \tag{F.1}$$

where $\hat{W}_t$ is a Wiener process under the risk-neutral probability, and ϕ, $\bar{r}$ and σ_v are three positive constants. The price of a zero predicted by this model is $P_t(r_t; T) = A(T-t)\, e^{-B(T-t)r_t}$ for two functions A and B, with obvious notation. The forward swap rate predicted by this model is:

$$R_t(r_t; T_1, \cdots, T_n) = \frac{P_t(r_t; T_0) - P_t(r_t; T_n)}{\text{PVBP}_t(r_t; T_1, \cdots, T_n)}, \tag{F.3}$$

the price value of the basis point is,

$$\text{PVBP}_t(r_t; T_1, \cdots, T_n) = \sum_{i=1}^{n} \delta_{i-1} P_t(r_t; T_i),$$

the instantaneous volatility of bond returns is,

$$\text{Vol-P}_t(r_t; T) \equiv -B(T-t)\, \sigma_v, \tag{F.3}$$

and, finally, the forward swap rate volatility is,

$$\begin{aligned} &\sigma_t(r_t; T_1, \cdots, T_n) \\ &= \frac{\text{Vol-P}_t(r_t; T)\, P_t(r_t; T) - \text{Vol-P}_t(r_t; T_n)\, P_t(r_t; T_n)}{P_t(r_t; T) - P_t(r_t; T_n)} - \frac{\sum_{i=1}^{n} \delta_{i-1} P_t(r_t; T_i)\, \text{Vol-P}_t(r_t; T_i)}{\text{PVBP}_t(r_t; T_1, \cdots, T_n)}. \end{aligned} \tag{F.4}$$

We simulate Eq. (F.1) using a Milstein approximation method, for initial values of the short-term rate in the interval $[0.01, 0.10]$. We use parameter values set equal to $\phi = 0.3807$, $\bar{r} = 0.072$ and $\sigma_v = \sqrt{0.02} \times 0.2341$, taken from Veronesi (2010, Chapter 15), and generate simulated values of $\sigma_t(r_t; T_1, \cdots, T_n)$, by plugging in the simulated values of r_t. We use the same parameter values for the Vasicek model in Eq. (F.1), with $\sigma_v = \sqrt{0.02}\sigma_r$. The two expectations,

$$\begin{aligned} &\mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(\text{PVBP}_T(r_T; T_1, \cdots, T_n) \int_t^T \sigma_s^2(r_s; T_1, \cdots, T_n)\, ds\right)\right] \\ &\text{and} \quad \mathbb{E}_Q\left[\int_t^T \sigma_s^2(r_s; T_1, \cdots, T_n)\, ds\right], \end{aligned} \tag{F.5}$$

are estimated through Montecarlo integration. The forward rates in Figure 7 and Table A.1 are obtained by plugging the initial values of the short-term rate into Eq. (F.3).

Table A.1

The IRS-VI$_n(t,T)$ index in Eq. (25) (labeled IRS-VI), and future expected volatility in a risk-neutral market, $\sqrt{\frac{1}{T-t}E_Q\left[V_n(t,T)\right]}$, where $V_n(t,T)$ is as in Eq. (A.6) (labeled E-Vol), for (i) time-to maturity $T-t$ equal to 1, 6 and 9 months, and 1, 2 and 3 years, and (ii) tenor length n equal to 1, 2, 3 and 5 years, and for time-to maturity $T-t$ equal to 1, 6 and 12 months, and (ii) tenor length n equal to 10, 20 and 30 years. The index and expected volatilities are computed under the assumption the short-term interest rate is as in the Vasicek model of Eq. (F.1), with parameter values $\phi = 0.3807$, $\bar{r} = 0.072$ and $\sigma_v = 0.0331$.

Maturity of the variance contract

Tenor = 1 year	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	2.18	124.83	124.82	2.48	106.53	106.53	3.27	73.50	73.52
3.00	3.80	71.97	71.97	4.01	66.39	66.38	4.53	53.39	53.40
5.00	5.44	50.62	50.62	5.54	48.25	48.25	5.80	41.93	41.94
7.00	7.08	39.08	39.08	7.08	37.93	37.93	7.08	34.54	34.55
9.00	8.73	31.85	31.86	8.63	31.27	31.27	8.36	29.38	29.38
10.00	9.56	29.17	29.17	9.41	28.78	28.75	9.00	27.34	27.35

Tenor = 2 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	2.93	78.12	78.12	3.53	60.11	60.12	3.84	52.74	52.76
3.00	4.31	53.66	53.66	4.70	45.49	45.50	4.91	41.59	41.61
5.00	5.68	40.93	40.93	5.88	36.62	36.62	5.98	34.36	34.37
7.00	7.07	33.12	33.12	7.07	30.66	30.67	7.06	29.29	29.30
9.00	8.47	27.84	27.84	8.26	26.39	26.40	8.15	25.53	25.55
10.00	9.18	25.80	25.80	8.86	24.68	24.69	8.69	24.01	24.02

Tenor = 3 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	3.51	55.76	55.76	4.02	45.18	45.20	4.28	40.52	40.54
3.00	4.68	42.23	42.23	5.02	36.50	36.51	5.20	33.69	33.71
5.00	5.87	34.04	34.04	6.04	30.64	30.65	6.12	28.85	28.87
7.00	7.06	28.55	28.55	7.06	26.44	26.45	7.05	25.25	25.27
9.00	8.27	24.62	24.62	8.08	23.26	23.27	7.98	22.47	22.49
10.00	8.87	23.04	23.04	8.60	21.95	21.96	8.45	21.30	21.32

Tenor = 5 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	4.31	34.63	34.62	4.69	29.56	29.58	4.89	27.16	27.18
3.00	5.20	29.11	29.11	5.46	25.75	25.77	5.59	24.05	24.07
5.00	6.11	25.17	25.17	6.23	22.84	22.86	6.30	21.61	21.63
7.00	7.04	22.20	22.20	7.02	20.55	20.56	7.01	19.64	19.66
9.00	7.97	19.89	19.89	7.82	18.70	18.71	7.74	18.01	18.03
10.00	8.44	18.92	18.92	8.22	17.90	17.91	8.11	17.30	17.32

Table A.1 continued

Maturity of the variance contract

Tenor = 1 year	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	3.61	64.25	64.26	4.67	41.87	41.92	5.38	31.66	31.72
3.00	4.76	49.10	49.10	5.45	36.06	36.10	5.92	28.90	28.96
5.00	5.91	39.72	39.72	6.24	31.66	31.69	6.46	26.58	26.63
7.00	7.07	33.35	33.35	7.04	28.22	28.25	7.00	24.61	24.65
9.00	8.23	28.75	28.75	7.83	25.46	25.48	7.54	22.91	22.95
10.00	8.82	26.90	26.90	8.23	24.27	24.29	7.81	22.15	22.19
Tenor = 2 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	4.13	47.30	47.32	5.01	32.95	33.01	5.61	25.69	25.76
3.00	5.10	38.61	38.62	5.68	29.27	29.32	6.07	23.86	23.92
5.00	6.07	32.62	32.62	6.35	26.34	26.38	6.52	22.27	22.33
7.00	7.05	28.25	28.25	7.02	23.94	23.97	6.98	20.89	20.94
9.00	8.04	24.92	24.92	7.69	21.94	21.97	7.44	19.66	19.71
10.00	8.53	23.53	23.54	8.03	21.07	21.10	7.67	19.11	19.15
Tenor = 3 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	4.52	36.92	36.94	5.28	26.87	26.93	5.79	21.41	21.48
3.00	5.36	31.47	31.49	5.85	24.41	24.46	6.18	20.14	20.21
5.00	6.19	27.44	27.45	6.42	22.37	22.41	6.57	19.02	19.08
7.00	7.04	24.33	24.34	7.00	20.64	20.69	6.97	18.02	18.08
9.00	7.89	21.87	21.88	7.58	19.17	19.21	7.36	17.12	17.17
10.00	8.31	20.82	20.83	7.87	18.51	18.55	7.56	16.70	16.76
Tenor = 5 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	5.07	25.19	25.22	5.65	19.35	19.41	6.04	15.86	15.93
3.00	5.71	22.66	22.68	6.09	18.10	18.16	6.34	15.19	15.26
5.00	6.36	20.60	20.61	6.53	17.01	17.06	6.64	14.58	14.64
7.00	7.01	18.90	18.91	6.98	16.05	16.10	6.95	14.02	14.08
9.00	7.67	17.47	17.48	7.43	15.20	15.25	7.26	13.50	13.56
10.00	8.00	16.83	16.84	7.65	14.81	14.86	7.41	13.26	13.31

Table A.1 continued

Maturity of the variance contract

Tenor = 10 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	5.25	17.80	17.80	5.48	15.92	15.93	5.72	14.12	14.14
3.00	5.81	16.51	16.51	5.97	14.96	14.97	6.13	13.43	13.45
5.00	6.39	15.42	15.41	6.47	14.13	14.14	6.54	12.82	12.84
7.00	6.98	14.48	14.48	6.97	13.40	13.41	6.96	12.27	12.29
9.00	7.59	13.67	13.67	7.49	12.75	12.77	7.39	11.78	11.80
10.00	7.91	13.31	13.31	7.76	12.46	12.47	7.61	11.55	11.57

Maturity of the variance contract

Tenor = 20 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	5.79	10.55	10.55	5.9561	9.65	9.66	6.11	8.74	8.76
3.00	6.17	10.27	10.27	6.2765	9.44	9.45	6.38	8.59	8.60
5.00	6.55	10.03	10.03	6.6067	9.24	9.26	6.65	8.44	8.46
7.00	6.95	9.79	9.79	6.9470	9.06	9.07	6.93	8.30	8.32
9.00	7.36	9.57	9.57	7.2978	8.88	8.90	7.22	8.17	8.18
10.00	7.57	9.47	9.47	7.4771	8.81	8.82	7.37	8.10	8.12

Maturity of the variance contract

Tenor = 30 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol	Forward Swap Rate	IRS-VI	E-Vol
1.00	5.96	8.70	8.70	6.0928	8.01	8.02	6.22	7.30	7.31
3.00	6.27	8.59	8.59	6.3651	7.93	7.93	6.45	7.23	7.25
5.00	6.60	8.49	8.49	6.6466	7.84	7.85	6.69	7.17	7.19
7.00	6.94	8.39	8.39	6.9374	7.76	7.77	6.93	7.11	7.13
9.00	7.29	8.29	8.29	7.2378	7.68	7.69	7.17	7.05	7.07
10.00	7.47	8.24	8.24	7.3917	7.65	7.66	7.30	7.02	7.04

The left panels of Figure 7 depict the approximations to the square root of the two expectations in Eq. (F.5): the first, normalized by $\mathrm{PVBP}_t\left(r_t; T_1, \cdots, T_n\right) \cdot (T-t)$, as in Eq. (25); and the second, normalized by $(T-t)$, as in Eq. (26). The right panels of Figure 7 depict the instantaneous volatility of the forward swap rate, as defined in Eq. (F.4). Finally, Eqs. (F.3), (F.4) and (F.5) are evaluated assuming reset dates are quarterly and, accordingly, setting δ_i constant and equal to $\frac{1}{4}$.

G. Spanning and hedging Gaussian contracts

G.1. Pricing

We price contracts where the variance payoff in Eq. (A.5) is replaced by:

$$V_n^{\mathrm{BP}}(t, T) \times \mathrm{PVBP}_T\left(T_1, \cdots, T_n\right),$$

where $V_n^{\mathrm{BP}}(t, T)$ denotes the realized variance of the forward swap changes in the interval $[t, T]$,

$$V_n^{\mathrm{BP}}(t, T)=\int_t^T R_s^2\left(T_1, \cdots, T_n\right) \sigma_s^2\left(T_1, \cdots, T_n\right) d s, \tag{G.1}$$

and $\sigma_s^2\left(T_1, \cdots, T_n\right)$ is as in Eq. (A.6). By a Taylor's expansion with remainder, we have, denoting, as usual, for simplicity, $R_t \equiv R_t\left(T_1, \cdots, T_n\right)$,

$$R_T^2=R_t^2+2 R_t\left(R_T-R_t\right)+2\left[\int_0^{R_t}\left(K-R_T\right)^{+} d K+\int_{R_t}^{\infty}\left(R_T-K\right)^{+} d K\right]. \tag{G.2}$$

Multiplying both sides of this equation by $\mathrm{PVBP}_t\left(T_1, \cdots, T_n\right)$, and taking expectations under the Q_{swap} probability, leaves:

$$\begin{aligned} & \mathrm{PVBP}_t\left(T_1, \cdots, T_n\right) \mathbb{E}_{Q_{\text{swap}}}\left(R_T^2-R_t^2\right) \\ & =2\left[\int_0^{R_t} \mathrm{SWPN}_t^{\mathrm{R}}\left(K, T ; T_1, \cdots, T_n\right) d K+\int_{R_t}^{\infty} \mathrm{SWPN}_t^{\mathrm{P}}\left(K, T ; T_1, \cdots, T_n\right) d K\right]. \end{aligned} \tag{G.3}$$

Moreover, by Itô's lemma, and Eqs. (A.4) and (G.1),

$$\mathbb{E}_{Q_{\text{swap}}}\left(R_T^2-R_t^2\right)=\mathbb{E}_{Q_{\text{swap}}}\left[V_n^{\mathrm{BP}}(t, T)\right].$$

By replacing this expression into Eq. (G.3) yields:

$$\begin{aligned} & 2\left[\int_0^{R_t} \mathrm{SWPN}_t^{\mathrm{R}}\left(K, T ; T_1, \cdots, T_n\right) d K+\int_{R_t}^{\infty} \mathrm{SWPN}_t^{\mathrm{P}}\left(K, T ; T_1, \cdots, T_n\right) d K\right] \\ & =\mathrm{PVBP}_t\left(T_1, \cdots, T_n\right) \mathbb{E}_{Q_{\text{swap}}}\left(R_T^2-R_t^2\right) \\ & =\mathrm{PVBP}_t\left(T_1, \cdots, T_n\right) \mathbb{E}_{Q_{\text{swap}}}\left[V_n^{\mathrm{BP}}(t, T)\right] \\ & =\mathbb{E}_Q\left[e^{-\int_t^T r_s d s}\left(\mathrm{PVBP}_T\left(T_1, \cdots, T_n\right) V_n^{\mathrm{BP}}(t, T)\right)\right] \\ & \equiv \mathbb{F}_{\mathrm{var}, n}^{\mathrm{BP}}(t, T), \end{aligned} \tag{G.4}$$

where the last line follows by the Radon-Nikodym derivative defined in Eq. (A.3).

The basis point volatility in Eq. (27) can also be simplified using the Black's formulae in Eqs. (B.1) and (B.10) to compute $\mathbb{F}_{\mathrm{var}, n}^{\mathrm{BP}}(t, T)$ in Eq. (G.4), as follows:

$\mathrm{IRS}\text{-}\mathrm{VI}_n^{\mathrm{BP}}(t, T)$

$$= \sqrt{\frac{2}{T-t}\left[\sum_{i:K_i<R_t}\hat{Z}_t\left(R_t^n, T, K_i; (T-t)\cdot \mathrm{IV}_{i,t}^2\right)\Delta K_i + \sum_{i:K_i\geq R_t} Z_t\left(R_t^n, T, K_i; (T-t)\cdot \mathrm{IV}_{i,t}^2\right)\Delta K_i\right]}, \tag{G.5}$$

where Z_t and $\hat{Z}_t$ are as in Eqs. (B.1) and (B.10), R_t^n is the current forward swap rate for maturity T and tenor length $T_n - T$, $R_t^n \equiv R_t(T_1, \cdots, T_n)$, and, finally, $\mathrm{IV}_{i,t}$ denotes the time t implied percentage volatility for swaptions with strike equal to K_i.

G.2. Hedging

We provide the details leading to Table III only, as those for Table IV are nearly identical. By Itô's lemma:

$$\begin{aligned}
&\mathrm{PVBP}_T(T_1,\cdots,T_n)\, V_n^{\mathrm{BP}}(t,T) \\
&= -2\mathrm{PVBP}_T(T_1,\cdots,T_n)\int_t^T R_s(T_1,\cdots,T_n)\, dR_s(T_1,\cdots,T_n) \\
&+ \mathrm{PVBP}_T(T_1,\cdots,T_n)\left(R_T^2(T_1,\cdots,T_n) - R_t^2(T_1,\cdots,T_n)\right).
\end{aligned} \tag{G.6}$$

By Eq. (G.2), the second term on the right hand side is the payoff at T of a portfolio set up at t, which is: (a) long $2R_t(T_1, \cdots, T_n)$ units a fixed interest payer swap struck at $R_t(T_1, \cdots, T_n)$, and (b) long a continuum of out-of-the-money swaptions with weights $2dK$. It is the static position (ii) in Table III of the main text. By Eqs. (G.2), (G.3) and (G.4), and the definition of the Radon-Nikodym derivative in Eq. (A.3), its cost is $\mathbb{F}_{\mathrm{var},n}^{\mathrm{BP}}(t,T)$, which we borrow at t, to repay it back at T, as in row (iii) of Table III. The self-financed portfolio to be shorted, as indicated by row (i) of Table III, is obtained similarly as the portfolio in row (i) of Table I (see Appendix C.3), but with the portfolio,

$$\hat{\psi}_\tau M_\tau = \mathrm{PVBP}_\tau(T_1,\cdots,T_n)\left(\int_t^\tau 2R_s(T_1,\cdots,T_n)\, dR_s(T_1,\cdots,T_n) - 1\right),$$

replacing that in Eq. (C.8).

G.3. Constant Maturity Swaps

Consider, initially, the fair price of the payment of a Constant Maturity Swap (CMS) occurring at time $T_0 + \kappa$, and set, for simplicity, $S(T_0) \equiv R_{T_0}(T_1, \cdots, T_n)$. The current value of $S(T_0)$ to be paid at time $T_0 + \kappa$ is the same as the current value of $S(T_0)\, P_{T_0}(T_0 + \kappa)$ to be paid at T_0, such that:

$$\mathrm{cms}(t, T_0+\kappa) = \mathbb{E}_Q\left(e^{-\int_t^{T_0} r_u du} S(T_0)\, P_{T_0}(T_0+\kappa)\right) = P_t(T_0+\kappa)\,\mathbb{E}_{Q_{\mathrm{swap}}}\left(S(T_0)\frac{\mathcal{G}_{T_0}}{\mathcal{G}_t}\right), \tag{G.7}$$

where $\mathcal{G}_\tau \equiv \frac{P_\tau(T_0+\kappa)}{\mathrm{PVBP}_\tau(T_1,\cdots,T_n)}$. We calculate $\mathcal{G}_\tau$, by discounting through the reset times, $T_i - T_0$, using the flat rate formula, $P_{T_0}(T_i) = (1 + \delta S(T_0))^{-i}$, where δ is the year fraction between the reset times, such that,

$$\mathcal{G}_\tau \equiv \frac{P_\tau(T_0+\kappa)}{\mathrm{PVBP}_\tau(T_1,\cdots,T_n)} \approx \frac{(1+\delta S(\tau))^{-(T_0+\kappa-\tau)}}{\delta\sum_{i=1}^n (1+\delta S(\tau))^{-(T_i-\tau)}} = \frac{S(\tau)(1+\delta S(\tau))^{-\kappa}}{1-\frac{1}{(1+\delta S(\tau))^n}} \equiv G(S(\tau)). \tag{G.8}$$

The previous derivations closely follow those in Hagan (2003), and are provided for completeness. We now make the connection between the price of the CMS in Eq. (G.7) and the price of the IRV forward contract in BP of Definition IV-(a) in the main text. Replacing the approximation in Eq. (G.8) into Eq. (G.7) leaves:

$$\mathrm{cms}(t, T_0+\kappa)$$

$$
\begin{aligned}
&= P_t\left(T_0+\kappa\right)\mathbb{E}_{\text{swap}}\left(S\left(T_0\right)\frac{G\left(S\left(T_0\right)\right)}{G\left(R_t\right)}\right)\\
&\approx P_t\left(T_0+\kappa\right)\mathbb{E}_{\text{swap}}\left(S\left(T_0\right)\left(1+\frac{G'\left(R_t\right)}{G\left(R_t\right)}\left(S\left(T_0\right)-R_t\right)\right)\right)\\
&= P_t\left(T_0+\kappa\right)\mathbb{E}_{\text{swap}}\left(S\left(T_0\right)\right)+\text{PVBP}_t\left(T_1,\cdots,T_n\right)G'\left(R_t\right)\mathbb{E}_{Q_{\text{swap}}}\left[S\left(T_0\right)\left(S\left(T_0\right)-R_t\right)\right]\\
&= P_t\left(T_0+\kappa\right)R_t+G'\left(R_t\right)\text{PVBP}_t\left(T_1,\cdots,T_n\right)\mathbb{E}_{Q_{\text{swap}}}\left(S^2\left(T_0\right)-R_t^2\right)\\
&= P_t\left(T_0+\kappa\right)R_t+G'\left(R_t\right)\mathbb{F}_{\text{var},n}^{\text{BP}}\left(t,T_0\right),
\end{aligned}
$$

where the second line follows by a first order Taylor approximation of the function G about R_t, the third from the definition of $G\left(R_t\right)$, the fourth from the martingale property of the forward swap rate under Q_{swap}, $\mathbb{E}_{\text{swap}}\left(S\left(T_0\right)\right)=R_t$ and, finally, the fifth equality from Eq. (G.3) and Eq. (G.4). Eq. (18) of the main text follows by calculating and summing every single CMS payment, $\text{cms}\left(t,T_0+j\kappa\right)$, for $j=1,\cdots,N$.

As mentioned in Section 5.3 of the main text, it is well-known since at least Hagan (2003) and Mercurio and Pallavicini (2006) that CMS link to the entire skew. However, our representation of the price of a CMS in Eq. (18) quite differs from previous ones. Mercurio and Pallavicini (2006), for example, utilize spanning arguments different from ours. We explain the differences.

Consider a Taylor's expansion with remainder of the function $f\left(R_T\right)\equiv R_T^2$ about some point R_o,

$$
R_T^2=R_o^2+2R_o\left(R_T-R_o\right)+2\left[\int_0^{R_o}\left(K-R_T\right)^+dK+\int_{R_o}^{\infty}\left(R_T-K\right)^+dK\right]. \tag{G.9}
$$

Mercurio and Pallavicini (2006) consider the point $R_o=0$, such that, Eq. (G.9) collapses to

$$
R_T^2=2\int_0^{\infty}\left(R_T-K\right)^+dK,
$$

such that

$$
\mathbb{E}_{Q\text{swap}}\left(R_T^2\right)=\frac{2}{\text{PVBP}_t\left(T_1,\cdots,T_n\right)}\int_0^{\infty}\text{SWPN}_t^{\text{P}}\left(K,T;T_n\right)dK.
$$

Our approach differs as we take $R_o=R_t$ in Eq. (G.9), leading to Eq. (G.2) and, then, to,

$$
\begin{aligned}
\mathbb{E}_{Q\text{swap}}\left(R_T^2-R_t^2\right)&=2\mathbb{E}_{Q\text{swap}}\left[\int_0^{R_t}\left(K-R_T\right)^+dK+\int_{R_t}^{\infty}\left(R_T-K\right)^+dK\right]\\
&=\frac{2}{\text{PVBP}_t\left(T_1,\cdots,T_n\right)}\left[\int_0^{R_t}\text{SWPN}_t^{\text{R}}\left(K,T;T_n\right)dK+\int_{R_t}^{\infty}\text{SWPN}_t^{\text{P}}\left(K,T;T_n\right)dK\right].
\end{aligned}
$$

G.4. Numerical experiments

Table A.2 reports experiments relating to the calculation of BP volatility as referenced by the index $\text{IRS-VI}_n^{\text{BP}}\left(t,T\right)$ in Eq. (33), as well as future expected volatility in a risk-neutral market, as predicted by the Vasicek model, using the same parameter values as those in Appendix F.

Table A.2

The IRS-VI$_n^{\mathrm{BP}}(t,T)$ index in Eq. (33) (labeled IRS-VI$^{\mathrm{BP}}$), and future expected volatility in a risk-neutral market, $\sqrt{\frac{1}{T-t}E_Q[V_n^{\mathrm{BP}}(t,T)]}$, where $V_n^{\mathrm{BP}}(t,T)$ is as in Eq. (G.1) (labeled E-Vol$^{\mathrm{BP}}$), for (i) time-to maturity $T-t$ equal to 1, 6 and 9 months, and 1, 2 and 3 years, and (ii) tenor length n equal to 1, 2, 3 and 5 years, and for time-to maturity $T-t$ equal to 1, 6 and 12 months, and (ii) tenor length n equal to 10, 20 and 30 years. The index and expected volatilities are computed under the assumption the short-term interest rate is as in the Vasicek model of Eq. (F.1), with parameter values $\phi = 0.3807$, $\bar{r} = 0.072$ and $\sigma_v = 0.0331$.

Maturity of the variance contract

Tenor = 1 year	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$
1.00	2.18	272.43	272.43	2.90	252.92	252.97	3.27	242.33	242.41
3.00	3.80	273.79	273.79	4.28	253.99	254.05	4.53	243.26	243.36
5.00	5.44	275.15	275.16	5.68	255.07	255.13	5.80	244.19	244.30
7.00	7.08	276.52	276.54	7.08	256.15	256.22	7.08	245.13	245.26
9.00	8.73	277.90	277.92	8.49	257.23	257.32	8.36	246.07	246.21
10.00	9.56	278.59	278.62	9.19	257.78	257.87	9.00	246.54	246.69

Tenor = 2 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$
1.00	2.93	229.60	229.60	3.53	213.30	213.37	3.84	204.45	204.55
3.00	4.31	231.15	231.15	4.70	214.53	214.60	4.91	205.51	205.63
5.00	5.68	232.71	232.72	5.88	215.76	215.84	5.98	206.57	206.72
7.00	7.07	234.29	234.30	7.07	217.00	217.09	7.06	207.65	207.81
9.00	8.47	235.88	235.89	8.26	218.25	218.35	8.15	208.73	208.90
10.00	9.18	236.68	236.69	8.86	218.87	218.99	8.69	209.28	209.46

Tenor = 3 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$
1.00	3.51	196.13	196.14	4.02	182.41	182.48	4.28	174.93	175.05
3.00	4.68	197.97	197.97	5.02	183.85	183.93	5.20	176.18	176.32
5.00	5.87	199.82	199.82	6.03	185.31	185.40	6.12	177.45	177.60
7.00	7.06	201.68	201.69	7.05	186.78	186.88	7.05	178.72	178.90
9.00	8.27	203.57	203.58	8.08	188.26	188.38	7.98	180.01	180.20
10.00	8.87	204.52	204.53	8.60	189.01	189.13	8.45	180.66	180.85

Tenor = 5 years	1 month			6 months			9 months		
Short Term Rate	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$	Forward Swap Rate	IRS-VI$^{\mathrm{BP}}$	E-Vol$^{\mathrm{BP}}$
1.00	4.31	149.31	149.31	4.69	139.19	139.27	4.89	133.66	133.79
3.00	5.20	151.57	151.58	5.46	140.98	141.07	5.59	135.22	135.36
5.00	6.11	153.87	153.87	6.23	142.79	142.90	6.30	136.80	136.96
7.00	7.03	156.20	156.21	7.02	144.63	144.75	7.01	138.39	138.57
9.00	7.97	158.56	158.57	7.82	146.49	146.61	7.74	140.01	140.20
10.00	8.44	159.76	159.76	8.22	147.43	147.56	8.11	140.82	141.02

Table A.2 continued

Maturity of the variance contract

Tenor = 1 year	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	3.61	232.52	232.62	4.67	199.65	199.91	5.38	174.98	175.33
3.00	4.76	233.32	233.44	5.45	200.12	200.39	5.92	175.25	175.62
5.00	5.91	234.13	234.27	6.24	200.58	200.88	6.46	175.53	175.91
7.00	7.07	234.95	235.10	7.04	201.05	201.36	7.00	175.81	176.21
9.00	8.23	235.77	235.93	7.83	201.53	201.85	7.54	176.09	176.50
10.00	8.82	236.18	236.35	8.23	201.76	202.10	7.81	176.23	176.64

Maturity of the variance contract

Tenor = 2 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	4.13	196.23	196.36	5.01	168.67	168.95	5.61	147.95	148.30
3.00	5.10	197.16	197.30	5.68	169.21	169.51	6.07	148.26	148.64
5.00	6.07	198.09	198.25	6.35	169.74	170.07	6.52	148.58	148.97
7.00	7.05	199.03	199.20	7.02	170.29	170.62	6.98	148.91	149.30
9.00	8.04	199.98	200.16	7.69	170.83	171.19	7.44	149.23	149.64
10.00	8.53	200.45	200.64	8.03	171.11	171.47	7.67	149.39	149.81

Maturity of the variance contract

Tenor = 3 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	4.52	167.99	168.13	5.28	144.62	144.91	5.79	127.00	127.35
3.00	5.36	169.09	169.24	5.85	145.26	145.57	6.18	127.38	127.75
5.00	6.19	170.19	170.36	6.42	145.90	146.23	6.57	127.76	128.14
7.00	7.04	171.31	171.49	7.01	146.55	146.89	6.97	128.14	128.54
9.00	7.89	172.43	172.63	7.58	147.20	147.56	7.36	128.53	128.93
10.00	8.31	172.99	173.20	7.87	147.53	147.89	7.56	128.73	129.13

Maturity of the variance contract

Tenor = 5 years	1 year			2 years			3 years		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	5.07	128.51	128.66	5.65	111.04	111.33	6.04	97.75	98.09
3.00	5.71	129.88	130.04	6.09	111.84	112.14	6.34	98.23	98.58
5.00	6.36	131.25	131.43	6.53	112.64	112.96	6.64	98.71	99.07
7.00	7.01	132.65	132.84	6.98	113.45	113.79	6.95	99.20	99.56
9.00	7.67	134.06	134.26	7.43	114.27	114.62	7.26	99.69	100.06
10.00	8.00	134.77	134.97	7.65	114.68	115.04	7.41	99.93	100.31

Table A.2 continued

Maturity of the variance contract

Tenor = 10 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	5.25	93.49	93.50	5.48	87.58	87.65	5.7299	81.26	81.39
3.00	5.81	96.00	96.00	5.97	89.57	89.65	6.1350	82.78	82.93
5.00	6.39	98.56	98.56	6.47	91.60	91.69	6.5477	84.33	84.49
7.00	6.98	101.18	101.19	6.97	93.67	93.77	6.9682	85.91	86.07
9.00	7.59	103.86	103.87	7.49	95.78	95.89	7.3965	87.51	87.68
10.00	7.90	105.23	105.23	7.76	96.85	96.96	7.6137	88.32	88.49

Maturity of the variance contract

Tenor = 20 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	5.7998	61.21	61.21	5.95	57.56	57.62	6.1146	53.62	53.72
3.00	6.1705	63.43	63.43	6.27	59.33	59.40	6.3827	54.99	55.10
5.00	6.5546	65.72	65.72	6.60	61.15	61.23	6.6577	56.38	56.50
7.00	6.9526	68.09	68.09	6.94	63.02	63.10	6.9396	57.80	57.93
9.00	7.3649	70.53	70.53	7.29	64.94	65.03	7.2287	59.25	59.39
10.00	7.5766	71.77	71.78	7.47	65.92	66.01	7.3759	59.99	60.13

Maturity of the variance contract

Tenor = 30 years	1 month			6 months			12 months		
Short Term Rate	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}	Forward Swap Rate	IRS-VI^{BP}	E-Vol^{BP}
1.00	5.96	51.91	51.91	6.09	48.88	48.93	6.22	45.59	45.68
3.00	6.27	53.95	53.95	6.36	50.51	50.57	6.45	46.85	46.94
5.00	6.60	56.06	56.07	6.64	52.19	52.25	6.69	48.13	48.24
7.00	6.94	58.25	58.25	6.93	53.91	53.98	6.93	49.45	49.56
9.00	7.29	60.51	60.51	7.23	55.69	55.77	7.17	50.79	50.91
10.00	7.47	61.66	61.66	7.39	56.59	56.68	7.30	51.47	51.60

G.5. Constant vega in Gaussian markets

We show that the statement in (17) is true—constant vega requires swaptions to be equally weighted in a Gaussian market. If the forward swap rate is solution to Eq. (13), the price of a swaption payer is:

$$\mathcal{O}_t^{\mathrm{P}}\left(R_t\left(T_1,\cdots,T_n\right),K,T,\sigma_N;T_n\right)\equiv \mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\mathcal{Z}_t^{\mathrm{P}}\left(R_t\left(T_1,\cdots,T_n\right),K,T,\sigma_N;T_n\right),$$

where

$$\mathcal{Z}_t^{\mathrm{P}}\left(R,K,T,\sigma;T_n\right)=\left(R-K\right)\Phi\left(\frac{R-K}{\sigma\sqrt{T-t}}\right)+\sigma\sqrt{T-t}\phi\left(\frac{R-K}{\sigma\sqrt{T-t}}\right),$$

and ϕ denotes the standard normal density. The vega of a payer is the same as the vega of a receiver, and equals:

$$\nu_t^{\mathcal{O}}\left(R,\sigma\right)\equiv\frac{\partial\mathcal{O}_t^{\mathrm{P}}\left(R,K,T,\sigma;T_n\right)}{\partial\sigma}=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\sqrt{T-t}\phi\left(\frac{R-K}{\sigma\sqrt{T-t}}\right),$$

such that the vega of a portfolio of swaptions (be they payers and/or receivers) is:

$$\nu_t\left(R,\sigma\right)\equiv\frac{\partial\pi_t\left(R,\sigma\right)}{\partial\sigma}=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\sqrt{T-t}\int\omega\left(K\right)\phi\left(\frac{R-K}{\sigma\sqrt{T-t}}\right)dK. \tag{G.10}$$

As for the if part in (17), let $\omega\left(K\right)=\text{const.}$, such that by Eq. (G.10), and the obvious fact that the Gaussian density ϕ integrates to one, we have indeed that the vega is independent of R,

$$\nu_t\left(R,\sigma\right)=\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)\sqrt{T-t}\cdot\text{const.}.$$

As for the only if part, let us differentiate $\nu_t\left(R,\sigma\right)$ in Eq. (G.10) with respect to R,

$$\frac{\partial\nu_t\left(R,\sigma\right)}{\partial R}=-\frac{\mathrm{PVBP}_t\left(T_1,\cdots,T_n\right)}{\sigma^2\sqrt{T-t}}\int\omega\left(K\right)\phi\left(\frac{R-K}{\sigma\sqrt{T-t}}\right)\left(R-K\right)dK.$$

The only function ω which is independent of R, and such that $\frac{\partial\nu_t(R,\sigma)}{\partial R}$ is identically zero, is the constant weighting.

H. Jumps

We derive the fair value of the Standardized IRV swap rate in Definition III, and the Standardized BP-IRV swap rate of Definition IV-(c), under the assumption that the forward swap rate follows the jump-diffusion process in Eq. (34). These derivations lead to the two indexes in Eq. (39) (percentage) and Eq. (41) (basis point). Section H.1 develops the arguments applying to the percentage contract and index, and Section H.2 contains derivations relating to the basis point.

H.1. Percentage

We apply Itô's lemma to Eq. (34), obtaining,

$$\begin{aligned}
d\ln R_\tau\left(T_1,\cdots,T_n\right)&=-\left(\mathbb{E}_{Q_{\mathrm{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta\left(\tau\right)\right)d\tau-\frac{1}{2}\left\|\sigma_\tau\left(T_1,\cdots,T_n\right)\right\|^2d\tau\\
&\quad+\sigma_\tau\left(T_1,\cdots,T_n\right)\cdot dW^*\left(\tau\right)+j_n\left(\tau\right)dN^*\left(\tau\right)\\
&=-\frac{1}{2}\left(\left\|\sigma_\tau\left(T_1,\cdots,T_n\right)\right\|^2d\tau+j_n^2\left(\tau\right)dN^*\left(\tau\right)\right)+\sigma_\tau\left(T_1,\cdots,T_n\right)\cdot dW^*\left(\tau\right)\\
&\quad-\left(\mathbb{E}_{Q_{\mathrm{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta\left(\tau\right)\right)d\tau+j_n\left(\tau\right)dN^*\left(\tau\right)+\frac{1}{2}j_n^2\left(\tau\right)dN^*\left(\tau\right).
\end{aligned}$$

We have,

$$\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)dN^*(\tau)=\left(\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta(\tau)\right)d\tau, \tag{H.1}$$

such that by the definition of $V_n^J(t,T)$ in Eq. (35), and Eq. (H.1), we obtain:

$$\begin{aligned}
&-2\mathbb{E}_{Q_{\text{swap},\tau}}\left(\ln\frac{R_T(T_1,\cdots,T_n)}{R_t(T_1,\cdots,T_n)}\right)-2\mathbb{E}_{Q_{\text{swap},\tau}}\left[\int_t^T\left(e^{j_n(\tau)}-1-j_n(\tau)-\frac{1}{2}j_n^2(\tau)\right)dN^*(\tau)\right]\\
&=\mathbb{E}_{Q_{\text{swap},\tau}}\left[V_n(t,T)\right]\\
&=\frac{1}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}\mathbb{E}_t\left[e^{-\int_t^T r_s ds}\mathrm{PVBP}_T(T_1,\cdots,T_n)V_n^J(t,T)\right]\\
&=\frac{\mathbb{F}_{\text{var},n}(t,T)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}\\
&=\mathbb{P}_{\text{var},n}^{J,*}(t,T),
\end{aligned} \tag{H.2}$$

where the third equality follows by the definition of the IRV forward agreement in Definition I, and the fourth by a straightforward generalization of Eq. (10). Comparing Eq. (H.2) with Eq. (10) and Eq. (C.2) produces Eq. (37).

H.2. Basis point

Apply Itô's lemma for jump-diffusion processes to Eq. (34), obtaining,

$$\begin{aligned}
&\frac{dR_\tau^2(T_1,\cdots,T_n)}{R_\tau^2(T_1,\cdots,T_n)}\\
&=-2\left(\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta(\tau)\right)d\tau+2\sigma_\tau(T_1,\cdots,T_n)\cdot dW^*(\tau)\\
&\quad+\|\sigma_\tau(T_1,\cdots,T_n)\|^2d\tau+\left(e^{2j_n(\tau)}-1\right)dN^*(\tau)\\
&=-2\left(\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta(\tau)\right)d\tau+2\left(e^{j_n(\tau)}-1\right)dN^*(\tau)+2\sigma_\tau(T_1,\cdots,T_n)\cdot dW^*(\tau)\\
&\quad+\|\sigma_\tau(T_1,\cdots,T_n)\|^2d\tau+\left(e^{j_n(\tau)}-1\right)^2dN^*(\tau).
\end{aligned} \tag{H.3}$$

By integrating, taking expectations under Q_{swap}, and using the definition of basis point variance, $V_n^{J,\text{BP}}(t,T)$ in Eq. (36), leaves:

$$\begin{aligned}
&\mathbb{E}_{Q_{\text{swap},t}}\left(R_T^2(T_1,\cdots,T_n)-R_t^2(T_1,\cdots,T_n)\right)\\
&=\underbrace{-2\int_t^T\left(\mathbb{E}_{Q_{\text{swap},\tau}}\left(e^{j_n(\tau)}-1\right)\eta(\tau)\right)d\tau+2\mathbb{E}_{Q_{\text{swap},t}}\left(\int_t^T\left(e^{j_n(\tau)}-1\right)dN^*(\tau)\right)}_{=0}\\
&+\underbrace{2\mathbb{E}_{Q_{\text{swap},\tau}}\left(\int_t^T\sigma_\tau(T_1,\cdots,T_n)\cdot dW^*(\tau)\right)}_{=0}\\
&+\mathbb{E}_{Q_{\text{swap},\tau}}[V_n^{J,\text{BP}}(t,T)],
\end{aligned}$$

where the first term is zero by Eq. (H.1). By Eq. (H.3), and Eq. (G.3), this cancellation leads to Eq. (40) and, hence, Eq. (41).

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