Volatility Analysis

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Outline

I. Historical Volatility

II. Fair Skew

III. PCA
Historical Volatility
Classical Estimator

\[ \sigma_{\text{historical}} = \sqrt{T} \sqrt{\frac{1}{N - 1} \sum_{i} (C_i - C_{i-1})^2} \]

\[ C_i = \log(\text{Close}_i) \]
Move-Based Estimator

For option pricing, a volatility estimate should reflect the cost of hedging.

Hedge according to spot moves (move based) better than hedge at fixed times (time based).
Commonly available data are Open, High, Low, and Close Prices
We denote the log of each by O, H, L, and C respectively
There are several estimates using the OHLC prices that improve upon the efficiency of the standard close-to-close estimator
\[ \sum (C_i - C_{i-1})^2 \]
Hi-Lo Estimates

- Parkinson:
  \[ \sigma_{park} = \sqrt{\frac{1}{N} \sqrt{\frac{1}{4(\ln 2)} \sum_{i=1}^{N} (H_i - L_i)^2}} \]

- Garman-Klass:
  \[ \sigma_{GK} = \sqrt{\frac{1}{N} \sqrt{\frac{1}{2} \sum_{i=1}^{N} (H_i - L_i)^2 - (2 \ln(2) - 1)(C_i - O_i)^2}} \]

- Rogers-Satchell (Drift Free Estimator):
  \[ \sigma_{RS} = \sqrt{\frac{1}{N} \sqrt{\sum_{i=1}^{N} [(H_i - C_i)(H_i - O_i) + (L_i - C_i)(L_i - O_i)]}} \]

- Yang-Zhang (Includes Overnight):
  \[ \sigma_{YZ} = \sqrt{\sigma_{close-to-open}^2 + k\sigma_{open-to-close}^2 + (1 - k)\sigma_{RS}^2} \]

\[ k = \frac{0.34}{1.34 + \frac{N + 1}{N - 1}} \]
The Uncertainty of the Night

- To compare with the classical estimate \( \sum (C_i - C_{i-1})^2 \), we should include overnight volatility \( \sum (O_i - C_{i-1})^2 \)
But…

□ High-Low Estimators are less useful than the classical one because they cannot be traded

□ Highs and Lows are always observed after the fact

□ However… we introduce a tradable estimator
Averaging Down

- Buy more of the stock every time it breaks its min:
  - At \( t \), hold \( S_0 - m_t \)
  - If no gap, average price = \( \frac{S_0 + m_t}{2} \)
  - \( P&L_t = Value_t - Cost_t \)

\[
= (S_0 - m_t)S_t - (S_0 - m_t)\left(\frac{S_0 + m_t}{2}\right)
\]

\[
= S_0S_t - m_tS_t - \frac{1}{2}[S_0^2 - m_t^2]
\]

\[
= (S_t - m_t)^2 - (S_0 - S_t)^2
\]

\[
= \frac{(S_t - m_t)^2}{2}
\]
Averaging Down

- Buy more of the stock every time it breaks its min:
  - At $t$, hold $S_0 - m_t$
  - If no gap, average price $P$:
    \[ P = \frac{S_0 + m_t}{2} \]

- \( P&L_t = Value_t - Cost_t \)
  \[ = (S_0 - m_t)S_t - (S_0 - m_t)\left(\frac{S_0 + m_t}{2}\right) \]
  \[ = (S_0 - m_t)\left[S_t - \frac{S_0 + m_t}{2}\right] \]
  \[ = (S_0 - m_t)^2 - (S_0 - S_t)^2 \]
A New Estimator

- From this we obtain a symmetrized estimator:

\[ \sigma_{\text{hist}}^2 = \sum_i \frac{(H_i - C_i)^2 + (C_i - L_i)^2}{2} \]

- We compare a lognormal version of this to the other estimators
Comparison of Estimators

SPX Index 30 Day Rolling Window Historical Volatility Estimate

- Historical Volatility
- Parkinson High-Low
- Garman-Klass Estimator
- Rogers-Satchell Estimator
- Yang-Zhang Estimator
- Tradable Estimator
- Tradable Estimator with Night

Date

Rolling Volatility

10.00  20.00  30.00  40.00  50.00  60.00  70.00  80.00
Fair Skew
P&L from Delta hedge
Can you guess the securities from their skews?

Crude Oil, Gold, S&P 500, VIX
Can you guess the securities from their skews?

S&P 500

VIX

Gold

Crude Oil
How correct is the market skew?

- Very simple test:
  - Buy an option of a strike $K$
  - Pay the premium
  - Delta hedge daily
  - Collect the pay-off if any
  - Compute P&L ($K$)

Premium and $\Delta$ are computed with the initial IV of this strike.

Repeat for a range of moneyness and average over several periods.
A Primer on Delta Hedging

\[ C(S, t) \]
A Primer on Delta Hedging

\[ C(S, t) \]

Call Price

Stock Price
A Primer on Delta Hedging

\[ C(S, t) \]

\[ C(S, t + \delta t) \]
A Primer on Delta Hedging

\[ \Delta P & L = \frac{\Gamma}{2} (\delta S)^2 - \theta \delta t = \frac{\Gamma}{2} S^2 \left( \left( \frac{\delta S}{S} \right)^2 - \sigma^2 \delta t \right) \]
Gold

P&L

Skew Maturity 0.25

Backtest Period Selector GLD US Equity 2012-09-10 To 2013-04-24
Gold – period right after
SPX

Backtest Period Selector SPX Index 2014-02-10 To 2014-06-27
Fair Skew

- For securities with illiquid or no options market what should be the fair value of options?

- How can we obtain the volatility surface for a security from just the historical time series of prices?

- Can fair skew computation be useful even for securities where options are available?
Fair Skew

- If we can compute the fair skew, we can answer the following questions
  - How to efficiently implement historical/implied vol arbitrage
  - Are the strong index equity skews justified?
  - What should be the pattern of implied volatilities for options on illiquid currency pairs?
  - Same question for swaptions, options on commodities spreads, etc
Fair Skew Methods

- Break Even Volatility
  - Delta Hedge
  - Gamma Weighted Average

- From Local Vols
  - Using forward PDE
  - Using harmonic mean approach
Break Even Volatility (BEV)

- If we sell an option for a premium corresponding to some volatility

- And perform delta-hedge computed with the same volatility

- We end up with a profit and loss that depends on $\sigma$ : $\text{P&L}(\sigma)$

- The $\sigma$ that sets this P&L to zero is called the Break-Even Volatility
P&L from delta hedging is given by

\[ P&L(\sigma) = (c_N - c_0) - \sum_{i=1}^{N} \Delta_{i-1}(S_i - S_{i-1}) \]

- \( N \) = number of time steps
- \( c_N \) = call price at maturity
- \( c_0 \) = Black-Scholes call price at time 0
- \( \Delta \) = Black-Scholes delta

Should be computed using root-finding algorithm or fixed point iterations
BEV2 – Gamma Weighted Average

- P&L from the continuously delta hedged option is given by

\[ \int_{0}^{t} \frac{1}{2} \Gamma \sigma S_{u}^{2} \left( \sigma_{u}^{2} - \sigma_{imp}^{2} \right) du \]

- where \( \sigma_{t} \) is the instantaneous realized volatility at time \( t \)

- So we need to solve for \( \sigma^{*} \) in the below equation

\[ \sum_{i} \frac{1}{2} \Gamma_{i} \sigma_{i} S_{i}^{2} \left( \left( \frac{\delta S_{i}}{S_{i}} \right)^{2} - \sigma^{2} \delta t \right) = 0 \]

- \( \Gamma_{i} \) in the above equation is a function of \( \sigma \) (hence need to solve it iteratively)
Solution to the above equation:

\[ \sigma^2 \delta t = \frac{\sum_i \frac{1}{2} \Gamma_i \sigma_i \left( \frac{\delta S_i}{S_i} \right)^2 \Gamma_i S_i^2}{\sum_i \Gamma_i S_i^2} \]

- BEVs for different strikes are averages of the same returns but weighted with different schemes.

- The weighting scheme gives more importance to the returns corresponding to a price level close to the strike.
BEV2 visualization

- BEV for one strike is linked to
  - the quadratic average of the returns (vertical peaks)
  - weighted by the gamma of the option (surface with the grid) corresponding to that strike.
BEV2 visualization

- We can perform the same operation on a set of windows to get a more complete and accurate picture
Fair Skew from Local Vols

- Local Vol can be obtained from the conditional expectation of the squared returns given $S_t$:

$$\frac{dS_t}{S_t} = \sigma(S_t, t)dW_t$$

$$\mathbb{E} \left( \left( \frac{dS_t}{S_t} \right)^2 \mid S_t = S \right) = \sigma^2(S, t)dt$$

- Conditional Expectation can be done by fitting a quadratic or spline fit.

- By taking the square root and annualizing we obtain the local vol estimate.
Fair Skew from Local Vols

- Short dated implied vol can be obtained as harmonic mean of the local vol as

\[
\sigma^{imp}(K) = \frac{\log \left( \frac{K}{S_0} \right)}{\int_{S_0}^{K} \frac{du}{u \sigma_{lv}(u)}}
\]

- Implied vol can also be obtained using the Dupire forward PDE to get call prices and then inverting them to get the implied vols

\[
\frac{\partial C}{\partial T} = \frac{1}{2} K^2 \sigma^2_{lv}(K, T) \frac{\partial^2 C}{\partial K^2}
\]
- Procedures described for a single time window ($S_i$ to $S_{i+D}$ where $D$ is time to maturity in days)
- For different time series windows we may get substantially different results
BEV-Time Window Aggregation

- Natural approach is to simply average the vols for each strike over the time windows.

- Alternative is to solve for the volatility that would have zeroed the average of the P&Ls over the different time windows.

- The second approach seems to yield smoother results.
SPX 3M Skew

Graph showing the SPX index from January 2011 to January 2015. The graph indicates an upward trend over the period.

Fair Skew

Graph showing the Fair Skew with data points for Local Vol Harmonic Mean and Implied Vol. The maturity in days is 63.
IBM 1M Skew
IBM 3M Skew

The image shows a graph titled "IBM 3M Skew". The graph plots the behavior of IBM US Equity from January 2011 to January 2015. The chart includes a secondary y-axis indicating "Fair Skew" with two curves: one labeled "Local Vol Harmonic Mean" and another labeled "Implied Vol". The x-axis represents moneyness, ranging from 60% to 150%, and the y-axis represents the fair skew percentage, ranging from 0% to 80%. The maturity in days is set to 63.
GOLD 1M Skew

Graph showing the skew for GLD US Equity from 2010 to 2015. The graph indicates fluctuations in the skew over time, with a peak around 2012. Below, a second graph illustrates the fair skew, comparing local vol harmonic mean and implied vol over moneyness from 60% to 150%. The maturity in days is set to 21 days.
Chinese Equity Options

- The Shanghai Stock Exchange will begin offering options on China 50 ETF (BBG ticker – 510050 CH Equity) from February 9.

- China 50 ETF tracks 50 of the largest companies -- mostly financial and resources firms -- on the Shanghai exchange.

- How do we set the prices on these options?
China 50 ETF Fair skews
Principal Component Analysis
PCA pre crisis
PCA post crisis
S&P 100 Constituents

Information Technology
- AAPL
- FB
- GOOG
- MA
- MSFT
- ABBV
- ABT
- GILD
- JNJ
- UNH
- AIG
- ACN
- EMR
- RGL
- Y
- AMGN
- LLY
- PFE
- ALL
- CSCO
- EBAY
- HPQ
- IBM
- QCOM
- TXN
- BAX
- BBB
- MDT
- MRK
- AXP

Health Care
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Consumer Discretionary
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S&P 100 July 2014 - Feb 2015
Returns PCA Factor 1. Variance explained: 42%
S&P 100 July 2014 - Feb 2015

Returns PCA Factor 2. Variance explained: 11%
S&P 100 July 2014 - Feb 2015
3M ATM Imp. Vol. PCA Factor 1. Variance explained: 56%

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IV Factor 1

[Heatmap with color gradient indicating variance explained: 4.00% to 18.00%]
S&P 100 July 2014 - Feb 2015
S&P 100: July 2014 - Feb 2015

Returns PCA. Variance explained: Factor 1: 42% Factor 2: 11%

3M ATM Implied Vol PCA. Variance explained: Factor 1: 56% Factor 2: 6%
S&P 100 July 2014 - Feb 2015
S&P 100 April 2011 - April 2012
S&P 100 April 2011 - April 2012
Returns PCA Factor 1. Variance explained: 64%
S&P 100 April 2011 - April 2012
Returns PCA Factor 2. Variance explained: 4%
S&P 100 April 2011 - April 2012

IV Factor:
6.00% 8.00% 10.00% 12.00% 14.00% 16.00% 18.00% 20.00% 22.00% 24.00% 26.00%
S&P 100 April 2011 - April 2012
3M ATM Imp. Vol. Factor 2. Variance explained: 5%
S&P 100 April 2011 - April 2012

Returns PCA Factor 2

3M ATM Impl. Vol. PCA Factor 2
S&P 100: April 2011 - April 2012

Returns PCA.

Variance explained: Factor 1: 64%  Factor 2: 4%

3M ATM Implied Vol PCA.  
Variance explained: Factor 1: 77%  Factor 2: 5%
S&P 100 April 2011 - April 2012
Thank You